Assignment I

Topics in Algorithms

- 1. There are two bins B_1 and B_2 . B_1 has 3 red balls and 4 green balls. B_2 has 2 red balls and 3 green balls. Suppose we toss a coin whose probability for head is $\frac{1}{4}$, pick a random ball from B_1 if the outcome is a head and pick a random ball from B_2 if the outcome is a tail.
 - 1. Write down a sample space for the experiment along with the probability of each point in the sample space.
 - 2. What is the probability of getting a red ball?
 - 3. Suppose you got a red ball in the end, what is the probability that the bin was B_2 ?
 - 4. Suppose you repeat the experiment a second time after removing the first ball that was picked from the respective bin, what is the probability that both the first ball and the second ball was red?
 - 5. Suppose both balls picked where red, what is the probability that on both occations the bin was B_1 ?
- 2. A 4 headed dice was thrown. Suppose the outcome was $1 < i \leq 4$, the experiment is repeated with a i 1 headed dice. (Thus tossing repeats till a 1 appears.)
 - 1. Write down the sample space modeling the experiment and the probabilities of each sample point.
 - 2. What is the probability of getting 3 on the first toss conditioned on the fact that the third toss was 1?
 - 3. What is the expected number of tossess required for the experiment to terminate?
 - 4. Conditioned on the event that the first throw gave a 4, what is the expected number of further throws required for the experiment to terminate?
- 3. Suppose there are k bins, and balls are being thrown uniformly at random into the bins one after another.
 - 1. What is the expected number of throws for reducing the number of non-empty bins below $\frac{k}{2}$?
 - 2. What is the expected number of throws for the first bin to have two balls?
 - 3. What is the expected number of throws necessary for having at least one bin carrying two balls? (That is, what is the expected number of insertions before the first collision in the hash table?)
 - 4. What is the expected number of throws for every bin to have at least one ball? (filling time?)
 - 5. What is the bound given by Markov's inequality on the number of throws required to reduce the probability of having at least one empty bin to be less than $\frac{1}{100}$?
- 4. Use integral upper bound to show that the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ is convergent.
- 5. Suppose $x_1, x_2, ..., x_n$ are numbers, Which among the quantities $(\frac{1}{n}(x_1 + x_2 + ..., x_n))^2$ and $\frac{1}{n}(x_1^2 + x_2^2 + ..., x_n^2)$ is larger? (Use Jensen's inequality). If X is a random variable, Show that $Var(X) = E(X^2) E^2(X) \ge 0$.
- 6. Suppose X and Y are independent random variables, then show that the random variables X E(X) and Y E(Y) are also independent. (linear shift by a constant value does not introduce new dependencies). Use this to give an alternate proof to show that when X, Y are independent, Var(X + Y) = Var(X) + Var(Y).
- 7. Suppose $X_1, X_2, ..., X_n$ are independent Berniolli random variables with $Pr(X_i = 1) = p$. Let $X = X_1 + X_2 ... + X_n$. Find E(X) and Var(X) in terms of p. Answer the question with $X_1, ..., X_n$ independent geometric random variables with $Pr(X_i - 1) = p$.
- 8. An array of n distinct elements is given. You pick a random element in the array and print it out iteratively. How may iterations are required (on the average) to print every element in the array at least once?
- 9. Suppose you have a graph with n vertices numbered 1, 2, 3, ..n. Suppose you re-label the vertices with a random permutation of 1, 2, ..n (that means the label of vertex i, label(i) is a number between 1 and n and each vertex gets a unique label between 1 and n with uniform probability). What is the expected number of vertices that has new label value equal to the original number?

Let X_i denote the random variable that takes value 1 if label(i) = i, 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$. Find the conditional expectation $E(X|X_1 = 1)$.