1. Let G be a directed graph. A subgraph G' of G is Eulerian if every vertex v in G' has $indegree(v) = outdegree(v)$. Show that the edges of every Eulerian subgraph G' can be decomposed into edge disjoint (directed) cycles.

2. The theorem of the last question will be re-phrased in the terminology of (linear) algebra in this and the next questions. Let G be a directed graph with n vertices, m edges and k components. Let F be a directed spanning forest in G. Let $T_1, T_2, ..T_k$ be (the unique) decomposition of F into maximal (directed) trees in G. Let E be the $n \times m$ edge vertex incidence matrix of G. $Nullspace(E)$ is called the *cycle space* of G.

    1. Show that for each $y \in \mathbf{R}^m$ such that y is the characteristic vector of a (directed) cycle in G, $Ey = 0$.
    2. Let e_i and e_j be two edges of G outside F. Let y_i and y_j be characteristic vectors in $\mathbf{R}^m$ corresponding to the *unique* cycles formed by adding edges e_i and e_j to F. Show that y_i and y_j are linearly independent in $\mathbf{R}^m$
    3. Show that the cycle space of G has dimension at least $m - n + k$.
    4. Suppose S is any subset of vertices that forms a connected component of G. Let x_S be the characteristic vector of the set set S in $\mathbf{R}^m$. show that $E^T x_S = 0$.
    5. Argue from the above that $Nullspace(E^T) \geq k$, or $Rank(E^T) = Rank(E) \geq n - k$.
    6. show that the cycle space of G has dimension exactly $m - n + k$.

3. Continuing with the notation in the previous question, let G be a graph with n vertices and m edges. Let E be the $n \times m$ incidence matrix of any arbitrary orientation of G. Let A be the adjacency matrix of G and D be the $n \times n$ diagonal matrix with $D(i,i) = deg(v_i)$, that is, the diagonal entries holding the degrees of the vertices. let $L = EE^T = A - D$ be he Laplacian of G.

    1. Show that $Nullity(E) = Nullity(EE^T)$. (This holds if E is any matrix, not necessarily an incidence matrix). What can you conclude about $Rank(L)$ from this?
    2. Let S be any subset of n edges of G. Let E_S be the $n \times n - 1$ matrix formed by picking the columns of E corresponding to the vertices in the set S. Show that E_S has rank $n - 1$ if and only if S forms the edges of a spanning tree of G.
    3. Suppose we remove a row from E_S to get an $(n-1) \times (n-1)$ matrix E'_S, show that $Rank(E_S) = Rank(E'_S)$.

4. Continuing with the notation in the previous questions, let G be a *connected* graph with n vertices and m edges. Let E be the $n \times m$ incidence matrix of an arbitrary orientation of G. Let T be a directed spanning tree in G. For any edge $e \in T$ removal of e from T disconnects T into exactly two components, with each vertex in G falling into exactly one of the components. All (directed) edges of G which connect vertices from one component to the other forms the *fundamental cut* defined by the edge e with respect to the tree T, denoted by C_e. Let $z_e \in \mathbf{R}^m$ be the characteristic vector of the set C_e.

    1. Let e, e' be two distinct edges of T, show that z_e and $z_{e'}$ are linearly independent.
    2. If y is the characteristic vector of any cycle in G, show that $y^T z_e = 0$ for each edge $e \in T$.
    3. Conclude that the cycle space of G has dimension at most $m - n + 1$. (This gives an alternate route for proving the dimension of the cycle space of a graph. Note that both approaches use the duality theorem).

5. Let A be an $n \times m$ matrix with $m \geq n$. Let $D = D(x_1, x_2, ..x_m)$ be the $m \times m$ diagonal matrix with variables $x_1, x_2, ..x_m$ on the diagonal. Consider the determinant $Det(ADA^T)$. Clearly, each term in the expansion has a monomial with n variables and there are ${}^mC_n$ terms in the expansion. For any subset S of n columns of A, let A_S be the $n \times n$ sub matrix of A formed out of picking the columns in S (in the order they appear in A).

    1. Consider the monomial term with the fewest number of variables. without loss of generality, let $x_1, x_2, ..x_r$, $r \leq n$ be the distinct variables appearing in the monomial. Show that the coefficient of the monomial is zero.
    2. Argue that every monomial term with number of distinct variables fewer than n will have coefficient zero.
    3. Let S be a subset of $\{1, 2, .., n\}$, $|S| = n$. Show that the coefficient of the monomial in the expansion of $Det(ADA^T)$ corresponding to the monomial $\prod_{i \in S} x_i$ is $Det(A_S)Det(A_S^T)$. (These sequence of questions prove the Cauchy Binut theorem correctly, for which the proof presented in the class had a mistake - what was the mistake?)

6. Give an example for:

    1. Non-planar Pfaffian orientable graph. (Give a Pfaffian orientation for the graph)
    2. A graph that is not Pfaffian orientable.

7. Given an $n \times n$ matrix A, define $permanent(A) = \sum_{\pi \in S_n} \prod_i a_{i,\pi(i)}$. Note that if we ignore the signs of the terms in the determinant expansion of A and consider each term positive, we get the permanent. Let $G(L, R, E)$ be an $n - n$ bipartite graph. Let $A = (a_{ij})$ be the $n \times n$ matrix formed with entries $a_{ij} = 1$ if there is an edge from vertex i in G to vertex j in R ($a_{ij} = 0$ otherwise); argue that the value of $permanent(A)$ is the number of perfect matchings in G.

8. Let A be an $n \times n$ real matrix. Let $r_i \in \mathbf{R}^m$ be the i^{th} row of A. Let α be any real number. Let $R(\alpha, i, j)$, $i \neq j$ denote the elementary row operation of multiplying the j^{th} row with α and adding this value to the i^{th} row. (Note that the j^{th} row is unchanged by this operation.) Show that $R(\alpha, i, j)$ is equivalent to multiplying A on the left with an $n \times n$ matrix whose determinant is 1. This proves that elementary row operations do not change the determinant. This observation will be used in solving the next question.

9. This question aims at proving the Cayley formula for counting the number of (labelled) trees over vertex set $\{1, 2, .., n\}$. using Kirchoff's matrix tree theorem. Let K_n be the complete graph of n vertices . Let A be the adjacency matrix and L be the Laplacian. Let L_i be the i^{th} minor of L. Note that L_i has -1 on all entries except on the diagonal where the values are $n - 1$.

    1. Argue that that $Det(L_i)$ evaluates to the number of trees of n vertices.
    2. Add all the rows of L_i to the first row. Now add the (new) first row to all the remaining rows. You will find that the determinant of this matrix is easy to evaluate. (Since all the above operations do not change the determinant (see previous question), you will get the Cayley formula from the determinant).
    3. Now we develop an alternate way to derive the same formula. First evaluate $Trace(L_i)$.
    4. Show that $L_i - nI$ is a matrix of rank 1. Hence argue that n is an Eigen value of L_i with multiplicity at least $n - 2$. Let λ be the remaining Eigen value.
    5. Now use the fact that $Trace(L_i)$ is the sum of its Eigen values and using the two expressions for trace, find λ.
    6. Finally use the fact that the determinant of a matrix is the product of its Eigen values (except for the sign) to derive the Cayley formula.

10. In this question, you will design a factor 2 approximation algorithm for the vertex cover problem. Given a graph $G(V, E)$ with a non-negative weight function w on the vertices. The problem is to find a minimum weight subset S of vertices such that every edge has atleast one endpoint in the set S.

    1. A weight function w is said to be *degree weighted* if there is a constant $c > 0$ such that $w(v) = c.deg(v)$ for each $v \in V$. if w is degree weighted and G connected, then show that the weight of *any* vertex cover in G is at most $2.OPT$ where OPT is the weight of the optimal vertex cover.
    2. Given an arbitrary weight function w, write a formula for the largest positive constant c such that $t(v) = c.deg(v) \leq w(v)$ for each vertex v.
    3. For the c formulated above, define residual weight $w'(v) = w(v) - t(v)$. When will a vertex have residual weight zero?
    4. Consider the following algorithm that takes as input a weighted graph G and returns a subset of vertices of G. If G has no edges return $\emptyset$. Otherwise, let D be the subset of vertices of G of degree zero and let R be the subset of vertices with residual weight zero. Remove all vertices in $D \cup R$ from G to get a residual graph G'. let S be the set returned when the algorithm recursively works on G', then return $R \cup S$. Show that the algorithm achieves a factor 2 approximation for the vertex cover problem.
    5. Find a graph on which the algorithm returns a vertex cover of weight twice the optimum value.