Assignment II

Computational Algebra

1. Let T be a linear transformation from a vector space V of dimension n to a vector space W of dimension m over the same field F.

    1. Show that $ker(T)$ is a subspace of V and $Img(T)$ is a subspace of W.
    2. Let $b_1, b_2, ..b_r \in V$ be chosen such that $T(b_1), T(b_2), .., T(b_r)$ is a basis of $Img(T)$. Show that $b_1, b_2, ..., b_r$ forms a linearly independent set in V.
    3. Let $u_1, u_2, ..u_k$ be a basis of $ker(T)$. Show that $\{b_1, b_2, .., b_r\} \cup \{u_1, u_2, ..u_k\}$ is a basis of V.
    4. Observe that $r = Rank(T)$ and $k = Nullity(T)$ (why?). Hence conclude that $Rank(T) + Nullity(T) = dim(V)$. This result is known as the **Rank Nullity Theorem**.

2. Let V be a vector space. Let $L(V)$ be the set of all linear transformations from V to F (that is scalar valued linear maps - these are called *linear functionals*). Fix any basis $\overline{b} = (b_1, b_2, ..., b_n)$ of V. Note that for any $l \in L$, the matrix of l with respect to basis $\overline{b}$ is the row vector $[l(b_1), l(b_2), ..., l(b_n)]$ (why?) Hence action on l on a vector v is computationally a dot product (except that no conjugation is needed) calculation (why?). Let $l_1, l_2, .., l_n$ be linear functionals defined by $l_i(b_j) = 1$ if $i = j$ and $l_i(b_j) = 0$ otherwise. Thus $l_1(b_1) = 1$, whereas $l_1(b_2) = l_1(b_3) = ..l_1(b_n) = 0$ and so on.

    1. Find the matrix representations of (that is, the row vectors of) $l_1, l_2, ..l_n$ with reference to the basis $b_1, b_2, ..b_n$.
    2. Let $l \in L$ be any linear functional. Show that we can find scalars $\alpha_1, \alpha_2, .., \alpha_n$ such that $l = \alpha_1 + \alpha_2 l_2 + ..\alpha_n l_n$. (Hint: The right side and the left side of the above expression are functions. To show that two functions are the same, what is needed is to show that for each input vector v, $l(v) = (\alpha_1 l_1 + \alpha_2 l_2 + ..\alpha_n l_n)(v)$. Let $v = x_1 b_1 + x_2 b_2 ... + x_n b_n$ be any arbitrary vector, show that if you can find scalars α_i such that the LS and RS are equal. Note that $l(b_1), l(b_2)..$ are scalars that does not depend on v.)
    3. Show that if $\beta_1 l_1 + \beta_2 l_2 + ..\beta_n l_n = 0$ the $\beta_1 = \beta_2 = ..\beta_n = 0$. (Hint: this too an expression involving functions on the LS and RS. Evaluate the LS on b_1 etc.)
    4. Conclude that L is a vector space of dimension n. $L(V)$ is sometimes the **dual space** of V. Given basis $b_1, b_2, .., b_n$ of V, the "corresponding basis" of $L(V)$, $l_1, l_2, ..l_n$ defined above is called the **dual basis** of $b_1, ..b_n$. Note that the definition of $l_1, ..l_n$ is dependent on $b_1, .., b_n$. (Intutively, once a basis is fixed for V, the cordinate vector of each $v \in V$ is an n entry column vector and each $l \in L$ is an $n-entry$ row vector. The dual basis $l_1, ..l_n$ defined above is simply the "standard basis" of this row space. Thus $L(V)$ can be thought of as the space of all row vectors).
    5. Let U be a subspace of V. Define $U^o = \{l \in L(V) : l(u) = 0 \forall u \in U\}$. (Once a basis if fixed, U^o is the set of all row vectors whose dot product with vectors in U is zero). Show that U^0 is a subspace of $L(V)$.
    6. Let U be a subspace of V of dimension k, Let $u_1, u_2, ..u_k$ be a basis of U. Extend the basis with vectors $u_{k+1}, ..u_n$ to form a basis of V. Define the dual basis $l_1, l_2, ..l_n$ such that $l_i(u_j) = \delta_{i,j}$. Show that $l_{k+1}, l_{k+2}, .., l_n$ is a basis of U^0. The result can be interpreted as saying that set of row vectors whose dot product with vectors in U evaluates to zero is a space of dimension $n - k$. This result is called **duality theorem**

3. Let A be an $m \times n$ matrix over some scalar field F. Consdier $ker(A) = \{x : Ax = 0\}$. By the Rank Nullity theorem, conclude that $Dim(ker(A)) = n - ColumnRank(A)$. Also observe that $ker(A) = RowSpace(A)^0$ (why?). Hence conclude using the Duality theorem that $Dim(ker(A)) = n - RowRank(A)$. From the two equalities, conclude that $ColumnRank(A) = RowRank(A)$. (We simply call the quantity $Rank(A)$).

4. Let U and W be subspaces of a vector space V such that $U \cap W = \{0\}$ and $dim(U) + dim(W) = dim(V)$. Show that $V = U \oplus W$. That is, V is a direct sum of U and W. Let $v \in V$. We know that there exists $u \in U, w \in W$ unique such that $v = u + w$. Define the linear transformation $P(v) = u$. (Basically $P(v)$ is the component along U in the representation of V as sum of U and W.)

   1. Show that $ker(P) = W$ and $Img(P) = U$.
   2. Show that $P^2 = P$.

5. Let V be an inner product space and let U be a subspace of V. Let $U^{\perp} = \{v \in V : (u, v) = 0\}$. We have seen in the class that $dim(U^{\perp}) = n - dim(U)$. (This is different from the dual space of the previous question). We have seen in class that any $v \in V$ can be written as $v = u + w$ for unique $u \in U$ and $w \in U^{\perp}$. The vector u is called the orthogonal projection of v on to the subspace U. In this question some properties of the projection.

   1. Let $u' \in U$. Show that $d(v, u') \geq d(v, u)$. That is, u is the point in U that is nearest to V. (Hint: Write $d^2(v, u') = ||v - u'||^2 = ||v - u + u - u'||^2 =< w + (u - u'), w + (u - u') >$ etc.)
   2. Define the $P(v) = u$. Note that P is well defined. Show that $Img(P) = U$, $ker(P) = W$, $P^2 = P$. show that P is Hermitian. Show that the Eigen values of P can be only be among $\{0, 1\}$. (orthongonal projections are Hermitian operators. But, not all Hermitian operators and orthogonal projections). The previous subquestion shows that the orthogonal projection is the closest approximation of a vector when restricted to a subspace.