Complementation of Büchi automaton

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In automata theory, **complementation of a Büchi automaton** is construction of another Büchi automaton that recognizes complement of the ω-regular language recognized by the given Büchi automaton. Existence of algorithms for this construction proves that the set of ω-regular languages and Büchi automata are closed under complementation.

This construction is particularly hard relative to the constructions for the other closure properties of Büchi automata. The first construction was presented by Büchi in 1962.[1] Later, other constructions were developed that enabled efficient and optimal complementation.[2][3][4][5]

Büchi's construction

Büchi presented[1] a doubly exponential complement construction in a logical form. Here, we have his construction in the modern notation used in automata theory. Let $A = (Q,\Sigma,\Delta,Q_0,\mathbf{F})$ be a Büchi automaton. Let $\sim_A$ be an equivalence relation over elements of Σ^+ such that for each $v,w \in \Sigma^+$, $v \sim_A w$ iff for all $p,q \in Q$, A has a run from p to q over v iff this is possible over w and furthermore A has a run via **F** from p to q over v iff this is possible over w. By definition, each map $f:Q \rightarrow 2^Q \times 2^Q$ defines a class of $\sim_A$. We denote the class by L_f. We interpret f in the following way. $w \in L_f$ iff, for each state $p \in Q$ and $(Q_1,Q_2)= f(p)$, w can move automaton A from p to each state in Q_1 and to each state in Q_2 via a state in **F**. Note that $Q_2 \subseteq Q_1$. The following three theorems provides a construction of the complement of A using the equivalence classes of $\sim_A$.

Theorem 1: $\sim_A$ has finitely many equivalent classes and each class is a regular language.

Proof: Since there are finitely many $f:Q \rightarrow 2^Q \times 2^Q$, $\sim_A$ has finitely many equivalent classes. Now we show that L_f is a regular language. For $p,q \in Q$ and $i \in \{0,1\}$, let $A_{i,p,q} = (\{0,1\}\times Q, \Sigma, \Delta_1\cup\Delta_2, \{(0,p)\}, \{(i,q)\})$ be a nondeterministic finite automaton, where $\Delta_1 = \{ ((0,q_1),(0,q_2)) \mid (q_1,q_2) \in \Delta\} \cup \{ ((1,q_1),(1,q_2)) \mid (q_1,q_2) \in \Delta\}$, and $\Delta_2 = \{ ((0,q_1),(1,q_2)) \mid q_1 \in \mathbf{F} \wedge (q_1,q_2) \in \Delta \}$. Let $Q' \subseteq Q$. Let $\alpha_{p,Q'} = \cap\{ L(A_{1,p,q}) \mid q \in Q'\}$, which is the set of words that can move A from p to all the states in Q' via some state in **F**. Let $\beta_{p,Q'} = \cap\{ L(A_{0,p,q})-L(A_{1,p,q})-\varepsilon \mid q \in Q'\}$, which is the set of non-empty words that can move A from p to all the states in Q' and does not have a run that passes through any state in **F**. Let $\gamma_{p,Q'} = \cap\{ \Sigma^+-L(A_{0,p,q}) \mid q \in Q'\}$, which is the set of non-empty words that can not move A from p to any of the states in Q'. By definitions, $L_f = \cap\{ \alpha_{p,Q2}\cap \beta_{p,Q1-Q2}\cap \gamma_{p,Q-Q1} \mid (Q_1,Q_2)=f(p) \wedge p \in Q\}$.

Theorem 2: For each $w \in \Sigma^\omega$, there are $\sim_A$ classes L_f and L_g such that $w \in L_f(L_g)^\omega$.

Proof: We will use infinite Ramsey theorem to prove this theorem. Let $w = a_0a_1...$ and $w(i,j) = a_i...a_{j-1}$. Consider the set of natural numbers **N**. Let equivalence classes of $\sim_A$ be the colors of subsets of **N** of size 2. We assign the colors as follows. For each i < j, let the color of {i,j} be the equivalence class in which w(i,j) occurs. Due to infinite Ramsey theorem, we can find infinite set X ⊆ **N** such that each subset of X of size 2 has same color. Let $0 < i_0 < i_1 < i_2 .... \in X$. Let f be a defining map of an equivalence class such that $w(0,i_0) \in L_f$. Let g be a defining map of an equivalence class such that for each j>0,$w(i_{j-1},i_j) \in L_g$. Therefore, $w \in L_f(L_g)^\omega$.

Theorem 3: Let L_f and L_g be equivalence classes of $\sim_A$. $L_f(L_g)^\omega$ is either subset of $L(A)$ or disjoint from $L(A)$.

Proof: Lets suppose word $w \in L(A) \cap L_f(L_g)^\omega$. Otherwise theorem holds trivially. Let r be the accepting run of A over input w. We need to show that each word $w' \in L_f(L_g)^\omega$ is also in $L(A)$, i.e., there exist a run r' of A over input w' such that states in **F** occurs in r' infinitely often. Since $w \in L_f(L_g)^\omega$, let $w_0w_1w_2... = w$ such that $w_0 \in L_f$ and for each i > 0, $w_i \in L_g$. Let s_i be the state in r after consuming $w_0...w_i$. Let I be a set of indices such that i ∈ I iff the run segment in r from s_i to s_{i+1} contains a state from **F**. I must be an infinite set. Similarly, we can split the word w'. Let $w'_0w'_1w'_2... = w'$ such that $w'_0 \in L_f$ and for each i > 0, $w'_i \in L_g$. We construct r' inductively in the following way. Let first state of r' be same as r. By definition of L_f, we can choose a run segment on word w'_0 to reach s_0. By induction hypothesis, we have a run on $w'_0...w'_i$ that reaches to s_i. By definition of L_g, we can extend the run along the word segment w'_{i+1} such that the extension reaches s_{i+1} and visits a state in **F** if i ∈ I. The r' obtained from this process will have infinitely many run segments containing states from **F**, since I is infinite set. Therefore, r' is an accepting run and $w' \in L(A)$.

Due to the above theorems, we can represent $\Sigma^\omega\text{-}L(A)$ as finite union of ω-regular languages of the from $L_f(L_g)^\omega$, where L_f and L_g are equivalence classes of $\sim_A$. Therefore, $\Sigma^\omega\text{-}L(A)$ is an ω-regular language. We can translate the language into a Büchi automaton. This construction is doubly exponential in terms of size of A.

References

1.

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