CS 6101 MFCS Test-I, Aug. 2017 Solution Key

1. Let $[1, 1]$ and $[1, -1]$ be the Eigen vectors of an operator T on $\mathbf{R}^2$ with Eigen values $+1$ and -1 respectively. What is the matrix of T with respect to the standard basis? Show calculation left side. 3

   Soln: Given $T(e_1 + e_2) = T(e_1) + T(e_2) = e_1 + e_2$ and $T(e_1 - e_2) = T(e_1) - T(e_2) = e_2 - e_1$. Adding the two equations, we get $2T(e_1) = 2e_2$ or $T(e_1) = e_2$. Substituting for $T(e_1)$ in the second equation we get $T(e_2) = e_1$. Thus we have:

   $[T(e_1), T[e_2)] = [e_1, e_2] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and hence the matrix of T w.r.t the standard basis is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. Suppose a real matrix A has 3 rows and 7 columns. Which among the following is true: 1) The first 4 columns of A cannot be linearly independent. 2) The first 4 columns of A may or may not be linearly independent (depends on the matrix). Give clear justification for your answer. 3

   Soln: The rank of A cannot exceed the row rank, which is at most 3 (why?). Hence, any collection of four or more columns must be linearly dependent (and cannot be linearly independent).

3. Let T be a linear operator $\mathbf{R}^4$, whose Eigen values are $1, 2, 3$ and 4 with corresponding Eigen vectors $[0001], [0011], [0111], [1111]$. Find $Nullity(T)$. Justify your answer. (Think!, don't start..). 3

   Soln: The matrix of the map w.r.t the basis of the given Eigen vectors is

   $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

   As A is non-singular T is bijective (why?). Consequently, $ker(T) = \{0\}$ and $Nullity(T) = 0$.

4. In $\mathbf{R}^3$, find the dual basis corresponding to the basis $[0, 0, 1], [0, 1, 1], [1, 1, 1]$. *Soln:* Let $b_1 = [0, 0, 1]^T, b_2 = [0, 1, 1]^T, b_3 = [1, 1, 1]^T$. Let $l_1[l_{11}, l_{12}, l_{13}], l_2 = [l_{21}, l_{22}, l_{23}], l_3 = [l_{31}, l_{32}, l_{33}]$ be the dual basis. We have $l_i.b_j = 1$ if $i = j$ and 0 otherwise. Solving the system of equations (actually this is nothing but a matrix inversion) we get $l_1 = [0, -1, 1]$, $l_2 = [-1, 1, 0]$ and $l_3 = [1, 0, 0]$ 3

5. Let T be a linear operator on a vector space V of dimension n. Let $b_1, b_2, \ldots, b_n$ be a basis of V. Suppose $T(b_1), T(b_2), \ldots, T(b_n)$ are linearly dependent, can we conclude that T is not injective? (Prove if true; give counter-example otherwise). Answer reverse side. 3

   Soln: Since $T(b_1), T(b_2), \ldots, T(b_n)$ are linearly dependent, there must exist scalars $x_1, x_2, \ldots, x_n$, not all zero, such that $x_1T(b_1) + x_2T(b_2) + \cdots x_nT(b_n) = T(x_1b_1 + x_2b_2 + \cdots + x_nb_n) = 0$. Since $b_1, b_2, \ldots, b_n$ is a basis, $x_1b_1 + x_2b_2 + \cdots + x_nb_n \neq 0$ (why?). Consequently, T is not injective.