CS 6101 MFCS - Test V, Sep.'17. Name:

Unless stated otherwise, assume V to be a complex inner product space with inner product (), $dim(V) = n$.

1. (3 points) Let A be the matrix of the inner product w.r.t basis $[b_1, b_2, \dots b_n]$. Let $[c_1, c_2, \dots c_n]$ be another basis such that $[b_1, b_2, \dots b_n] = [c_1, c_2, \dots c_n]B$. What will be the matrix of the inner product with respect to basis $[c_1, c_2, \dots c_n]$? (Answer in terms of A and B.)

   Soln: Let A' is the matrix of the inner product with respect to $[c_1, c_2, \dots c_n]$. If $\vec{x}, \vec{y}$ be coordinate vectors of $u, v \in V$ wrt $[b_1, b_2, \dots b_n]$, then $B\vec{x}, B\vec{y}$ will be their coordinates wrt $[c_1, c_2, \dots c_n]$. Hence, the inner product $(u, v) = (B\vec{x})^T A' \overline{B\vec{y}} = \vec{x}^T A \overline{\vec{y}}$. Since $\vec{x}, \vec{y}$ were chosen arbitrary, we have $A = B^T A' \overline{B}$ or $A' = (B^T)^{-1} A (\overline{B})^{-1}$.

2. (3 points) Consider the basis $[\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]^T, [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$ of $\mathbf{R}^2$. What is the matrix of the standard inner product w.r.t this basis? Justify your answer.

   Soln: Since $[\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]^T, [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$ is an orthonormal basis for $\mathbf{R}^2$ wrt the standard basis, the matrix will be the 2×2 identity matrix.

3. (3 points) What are the coordinates of the the vector $[1, 1, 1, 2]^T$ w.r.t the basis $b_1 = \frac{1}{2}[1, 1, 1, 1]^T, b_2 = \frac{1}{2}[1, -1, 1, -1]^T, b_3 = \frac{1}{2}[1, j, -1, -j]^T, b_4 = \frac{1}{2}[1, -j, -1, j]^T$ of $\mathbf{C}^4$? (Think!, Don't start calculation).

   Soln: Let $v = [1, 1, 1, 2]^T$. It is easy to see that $[b_1, b_2, b_3, b_4]$ is an orthonormal basis for $\mathbf{C}^4$. with respect to the standard inner product. Hence, $v = (v, b_1)b_1 + (v, b_2)b_2 + (v, b_3)b_3 + (v, b_4)b_4$. The coordinates are $[1, 1, 1, 2]\frac{1}{2}\overline{[1, 1, 1, 1]^T} = \frac{5}{2}$, $[1, 1, 1, 2]\frac{1}{2}\overline{[1, -1, 1, -1]^T} = -\frac{1}{2}$, $[1, 1, 1, 2]\frac{1}{2}\overline{[1, j, -1, -j]^T} = \frac{j}{2}$, $[1, 1, 1, 2]\frac{1}{2}\overline{[1, -j, -1, j]^T} = -\frac{j}{2}$.

4. (3 points) Find the point on the line $x + y = 0$ that is closest to the point $(1, 0)$ in $\mathbf{R}^2$ (with respect to Euclidean distance).

   Soln: $u = [\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$ is a unit vector along the line $x + y = 0$. Hence Projection of $[1, 0]^T$ on the direction defined by u gives the vector nearest to $[1, 0]^T$ along $x + y = 0$ with respect to the Euclidean distance. The projection is $(([1, 0][\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T)u = [\frac{1}{2}, -\frac{1}{2}]^T$

5. (3+3 points) Let $H : V \mapsto V$ be a linear map satisfying for all $u, v \in V$, $(u, Hv) = (Hu, v)$. Let λ be an Eigen value of H.

   1. Show that λ is a real number

      Soln: Let x be an Eigen vector corresponding to Eigen value λ. Then $\lambda||x||^2 = (\lambda x, x) = (Hx, x) = (x, Hx) = (x, \lambda x) = \overline{\lambda}||x||^2$. As $x \neq 0$, We have $||x|| \neq 0$ and hence $\overline{\lambda} = \lambda$, which implies that λ is real.

   2. Let $E_\lambda = \{v \in V : Hv = \lambda v\}$, the Eigen space associated with the Eigen value λ. Let $u \in E_\lambda^\perp$. Show that $Hu \in E_\lambda^\perp$. ($E_\lambda^\perp$ is the orthogonal compliment space of E_λ.)

      Soln: Let $v \in E_\lambda$ and $w \in E_\lambda^\perp$ be chosen arbitrarily. The choice of v, w ensures that $(v, w) = 0$. It is enough to prove that $(v, Hw) = 0$ (why?). But $(v, Hw) = (Hv, w) = (\lambda v, w) = (\lambda v, w) = \lambda(v, w) = 0$.