CS 2006: Discrete Structures **Final Part II** **April 2018**

1. Write down FOLG(=) formula to express the condition "every vertex has in-degree one and out-degree two". [2]

   Soln: Suppose $Degone(x)$ and $Degtwo(x)$ specify that x has in-degree one and out-degree two respectively. Then the required formula is: $\forall x(Degone(x) \wedge Degtwo(x))$. Now,
   $Degone(x) = \exists y[G(y,x) \wedge (\forall z G(z,x) \implies (z = y))]$.
   $Degtwo(x) = \exists y_1, \exists y_2[(y_1 \neq y_2) \wedge G(x,y_1) \wedge G(x,y_2) \wedge (\forall z G(x,z) \implies (x = y_1 \vee x = y_2))]$.

2. Does the condition above admit any infinite models? Justify your answer. [2]

   Soln: Consider the graph over the infinite vertex set $V = \{a, b, c, a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2 \ldots\}$.
   $E = \{(a,b),(b,c),(c,a),(a,a_0),(b,b_0),(c,c_0)\} \cup \{(a_i, a_{2i+1}) : i \geq 0\} \cup \{(a_i, a_{2i+2}) : i \geq 0\}$
   $\cup\{(b_i, b_{2i+1}) : i \geq 0\} \cup \{(b_i, b_{2i+2}) : i \geq 0\} \cup \{(c_i, c_{2i+1}) : i \geq 0\} \cup \{(c_i, c_{2i+2}) : i \geq 0\}$.

3. How many connected graphs with vertices labelled $\{1, 2, \ldots, n\}$ exist such that every vertex has degree two? [2]

   Soln: Connected graphs with every vertex having degree two are precisely cycles. Any circular permutation of $\{1, 2, \ldots, n\}$ yields a distinct labelling. However, each pair of clock-wise and anti-clockwise labelling yields the same graph. Hence, $\frac{|\underline{n-1}}{2}$.

4. How many graphs are possible with 17 vertices such that average degree is 3? [2]

   Soln: The total degree $17 \times 3 = 51$, which is impossible because the total degree has to be even (equals twice the number of edges). Hence no such graph exists.

5. Draw the labelled tree corresponding to the Prufer code $(3, 7, 4, 1, 5, 6, 2, 9)$. **Justify** your answer. [2]

   Soln: $V = \{0, 1, 2, \ldots, 9\}$. $E = \{(8,3),(3,7),(7,4),(4,1),(1,5),(5,6),(6,2),(2,9),(9,0)\}$

6. In $\mathbf{R}^2$, consider the ordering $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. Let $S = \{(x, y) : x^2 + y^2 = 1\}$. What are the minimal elements of the set? [2]

   Soln: The required set is $\{(x, y) : x^2 + y^2 = 1 \wedge \forall p \forall q[(p^2 + q^2 = 1) \implies (x \leq q \vee y \leq q)]\}$. This is precisely the collection of points on the unit circle in the third quadrant. That is,
   $\{(x, y) : (x \leq 0) \wedge (y \leq 0) \wedge (x^2 + y^2 = 1)\}$.

7. Let $(S\ \leq)$ be a complete lattice. Let $f : S \mapsto S$ satisfy $f(\sup(A)) = \sup(f(A))$ for every non-empty subset A of S. Let $A = \{0, f(0), f^2(0), \ldots\}$. Prove that $\sup(A)$ is a fix point of f. [2]

   Soln: If $f(0) = 0$, then $A = f(A) = \{0\}$ and $\sup(A) = 0 = \sup(f(A))$. Otherwise, $f(A) = A - \{0\}$ (why?). Note that UB(A) =UB$(f(A))$ (why?). Consequently, $\sup(A) = \sup(f(A))$ (why?). Then, by hypothesis, $f(\sup(A)) = \sup(f(A)) = \sup(A)$. Thus $\sup(A)$ is a fix point of f.

8. What is the maximum number of edges possible in a graph with $2n$ vertices without any odd cycles? **Justify** your answer. [3]

   Soln: A graph has no odd cycles if and only if it is bipartite. Let the partitions be of size t and $2n - t$. The maximum number of edges possible for this graph is $t(2n - t)$ (why?). The value of t that maximizes this expression is $t = n$ (differentiate w.r.t t ...). The number of edges corresponding to $t = n$ will be n^2.

9. Prove that every graph with at least two vertices contain two (or more) vertices of the same degree. [3]

   Soln: The claim is true for graphs with 2 vertices. Suppose that a graph has $n > 2$ vertices and is not connected, then if at least one connected component has two or more vertices, the claim follows by induction. If all connected components of the graph are single vertices, then also the claim holds (why?). Finally, if a graph has $n > 2$ vertices and is connected, then the degree of each vertex in the graph must be a number between 1 and $n - 1$. Since there are n vertices and $n - 1$ degree values, by pigeon hole principle, at least two vertices must have the same degree.