Department of Computer Science and Engineering, NIT Calicut

Lecture 0: Preliminaries on Graphs

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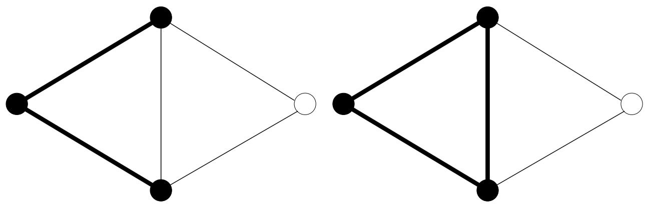
Basic Definitions[1]

These notes assume background in data structures and algorithms (see for example [1]) and discrete mathematics as in [3] as prerequisites. Standard notions are defined here only for fixing the notation.

A graph $G = (V, E)$ will be simple undirected unless stated otherwise. Normally we used n to denote the number of vertices and e (sometimes m) to denote the number edges in a graph G. Similarly we will simply use V and E for $V(G)$ and $E(G)$ when the meaning is clear from the context. We denote by $deg(v)$ the degree of a vertex. A graph is *regular* if all its vertices have the same degree

G' is a subgraph of G if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$ and the end points of edges in $E(G')$ are in $V(G')$. G' is an induced subgraph of G if it is a subgraph of G and it has all the edges of G whose end points are in $V(G')$.



(a) Subgraph but not induced (b) Induced subgraph

Figure 1: Darkened elements belong to subgraph

A *walk* in a graph is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has the endpoints v_{i-1} and v_i. A *path* is a walk without vertices repeating. If the end points of a path are connected by an edge, then the vertices of the path forms a *cycle*. G is *connected* if there is a path connecting every pair of vertices. A forest is a graph without a cycle. A *tree* is a connected graph without a cycle. A *component* in a graph G is a maximally connected subgraph. A subgraph G' of a graph G is called a *spanning tree* in G if G' is a tree and has n vertices.

Exercise 1. *Show that the sum of the degrees of each vertex in a graph equals twise the number of edges in G.*

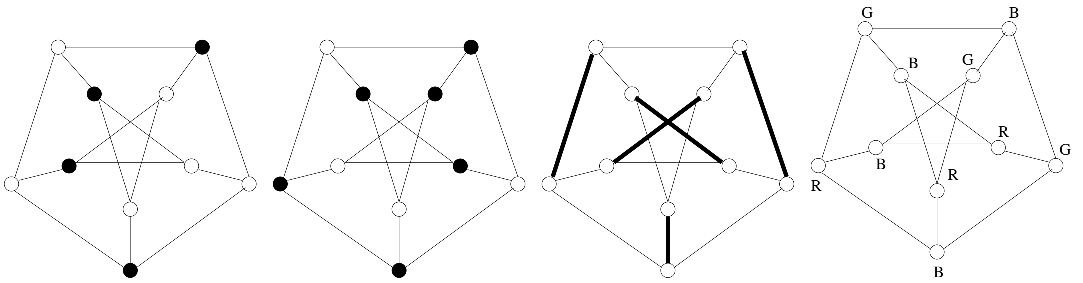
Exercise 2. *How many edges are there in a graph without a cycle with n vertices and k components?*

[1] We follow [4] for the terminology and fundamental concepts.

The complement graph $\overline{G}$ of a graph G is a graph with the vertex set $V(G)$ and all possible edges between vertices in $V(G)$ which are not there in G.

An *independent set* of a graph is a subset of its vertices such that there are no adjacent pairs of vertices in it. A *Matching* of a graph is a subset of its edges so that no two of which share a common vertex. A *vertex cover* of a graph is a subset of its vertices such that every edge has atleast one endpoint in it.

The *Clique number* of a graph is the size of the maximum complete subgraph of it whereas the *chromatic number* of a graph is the minimum number of colours required to properly colour (adjacent vertices get different colours) the vertices of it.



(a) Max. ind. set set (b) Min. vertex cover (c) Max. matching (d) Min. proper colouring

Figure 2: Various graph parameters are depicted using Petersen graph

Exercise 3. *Show that the size of the maximum matching in a graph G cannot exceeed the size of the minimum vertex cover. Can you find a similar relation between clique number and chromatic number?*

Exercise 4. *If max clique in G has size k what can you say about minimum vertex cover in $\overline{G}$? What can you say about the sizes of maximum independent set or chromatic number of $\overline{G}$?*

A graph is $k - partite$ if its vertex set can be partitioned into k (possibly empty) partitions such that each partition contains vertices with no edges in between. For example, *bipartite graphs* are graphs with two such partitions. A (proper) *colouring* of the vertices of a graph assigns colours to each vertex in G such that adjacent vertices get different colours. Clearly a graph is k colourable if and only if it is k partite and the chromatic number of G is the minimum k for which G is k partite. In particular bipartite graphs precisely form the class of two colourable graphs.

Bipartite, Eulerian and Hamiltonian Graphs

Bipartite graphs are characterized by the absence of odd cycles.

Theorem 1. *(König, 1936) A graph is bipartite if and only if it has no odd cycle.*

Proof. Assume that G is bipartite. Every walk in G alternates between the vertices in two partitions. So, there can not have an odd cycle in G. Conversely let G be a connected graph with no odd cycle. It suffices to show that G is two colourable.

Consider a BFS on G starting from any vertex v. Let L_i be the set of vertices at level i ($i \geq 0$) in the DFS tree (with v at level 0). It is the property of BFS that an edge at level i can have an edge only to vertices at level $i, i+1$ or $i-1$. However two vertices w and w' at level i cannot have an edge as the edge (w, w') along with the paths from u to w and u to w' will form an odd cycle. Now colouring every vertex at odd level RED and even level BLUE gives a two colouring of G. □

A closed walk with all the edges with no repetition is known as *Eulerian circuit.* A graph with an Eulerian circuit is known as *Eulerian graph.* There can only be atmost one non-trivial component for an Eulerian graph. Note that, in an Eulerian circuit each passage through a vertex uses two incident edges. It follows that the degree of each vertex in an Eulerian graph is even. Euler stated in 1736 that this is sufficient for a graph with atmost one non-trivial component.

Theorem 2. *A graph G is Eulerian if and only if it has atmost one nontrivial component and its vertices all have even degree.*

Proof. Aassume G is connected and vertices have even degree. Start a traversal from any vertex of G. Stop it when it meets a vertex already traversed. This is possible as all the vertex degrees are even for that component and hence there is an edge to leave any vertex to which we enter. Thus we get a cycle from that walk. Delete those edges which forms the cycle from the graph. We end up with possibly many non-trivial components having only even-degreed vertices. Recursively find a Eulerian circuit and attach it to the removed cycle to form an Eulerian circuit for G (Fig. 3). □

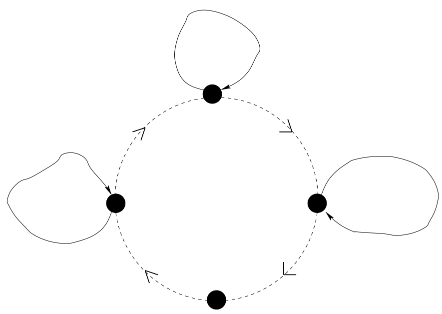


Figure 3: Eulerian circuit is formed from that of the components obtained inductively.

A similar concept is Hamiltonian cycle. A *Hamiltonian cycle* is a cycle with every vertex included. A graph with a Hamiltonian cycle is known as a *Hamiltonian graph.* Complete graphs are Hamiltonian and for a bipartite Hamiltonian graph, the size of both the partitions should be same. Though it is very easy to verify whether a graph has an Eulerian circuit or not, it is **NP** Complete to recognize a Hamiltonian graph.

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Lecture 2: Colouring Planar Graphs

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Graph (vertex) colouring refers to assigning colours to vertices of a graph such that adjacent vertices get different colours. The *Chromatic number,* $\chi(G)$ of a graph is the minimum number of colours required for a proper colouring of G. It is easy to see that the chromatic number of a graph G is nothing but the smallest k such that the graph is $k - paritite$. The problem of finding a minimum proper colouring of graph is **NP** hard.

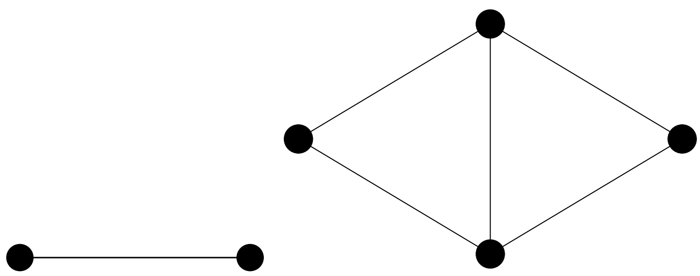
The *greedy colouring* of a graph G related to a vertex ordering $v_1, v_2, \ldots, v_n$ of G is obtained by colouring vertices in that order, assigning v_i the smallest indexed colour which is not used by its lower indexed neighbours.

Exercise 5. *Show that using greedy colouring we can colour a graph with* $\Delta(G) + 1$ *where* $\Delta(G)$ *is the degree of the vertex with the maximum degree in* G*. Show that the bound is tight for odd cycles and complete graphs.*

For another solution to the exercise, remove a vertex from the graph. Recursively colour the resultant graph with $\Delta(G)$ colours. Put back the vertex and we have atleast one free colour for it.

A graph is *planar* if it can be drawn on the plane without edges crossing. In this lecture, we prove that every planar graph is 5 colourable and also see how to find such a colouring.

Suppose G is connected planar. Then a drawing of G on the plane divides the plane into different disconnected regions called *faces.* Figure 4 illustrates planar graphs with one and three faces respectively. We will denote by n, e and f the number of vertices, edges and faces in a graph.



(a) $f = 1$ (b) $f = 3$

Figure 4: Planar graphs

Euler's Formula

Assume that we are given connected planar graph G drawn on the plane without edges crossing. Remove edges one by one from G till G is a tree. Let the resultant tree be called

T. Since T is a tree, T has only one face and $f = 1$. Since $e = n - 1$ for a tree, the relation,

$$f + n = e + 2 \tag{1}$$

holds.

Add each edge back into T. Addition of each edge creates a new face and both sides of the above equation increases by one. So the equation given above holds for any connected planar graph. Note that the argument holds for any drawing of G on the plane. Thus we have the theorem:

Theorem 3. *Let n, e and f be the number of vertices, edges and faces of a connected planar graph G then $f + n = e + 2$.*

The above characterisation of planar graphs was discovered by the famous mathematician *Leonhard Euler* [1707-1783].

Existence of a small degree vertex

For a planar graph G with at least three vertices, each face is surrounded by at least 3 edges. Hence, if we sum (over all faces) the number of edges around each face, we get at as sum at least $3f$. On the other hand each edge can occur at most twice in the sum as an edge can be shared by at most two faces. Hence, the above sum cannot exceed $2e$. Thus we have:

Lemma 1. *If G is a planar graph with at least three vertices, then $3f \leq 2e$.*

Multiplying (1) with 3 gives $3n + 3f = 3e + 6$. Substituting for $3f$ from the lemma in this equation yields $3n + 2e \geq 3e + 6$ or $e \leq 3n - 6$.

The sum of the degrees of all vertices in a graph is $2e$ as each edge contributes two to the sum. Hence, the average degree of a vertex in G is given by:

$$deg_{avg}(G) = \frac{\sum_{v \in V(G)} deg(v)}{n} = 2e/n. \tag{2}$$

Using the fact that $e \leq 3n - 6$ for a planar graph with $n \geq 3$ we get:

$$deg_{avg}(G) \leq \frac{2(3n-6)}{n} \leq 6 - \frac{12}{n} < 6 \tag{3}$$

We thus have the theorem:

Theorem 4. *If G is planar with at least 3 vertices, then there is a vertex in G of degree at most 5.*

This yields an immediate algorithm for six colouring planar graphs.

Theorem 5. *Every planar graph is six colourable.*

Proof. Let G be planar. If G has less than 7 vertices, the result is trivial. Otherwise, let v be a vertex satisfying $deg(v) \leq 5$ (see previous theorem). Remove v from G. The resultant graph G' is planar. Inductively do a six colouring of each component of this graph and put back v. Since v has at most five neighbours in G', we will be left with at least one free colour for v. □

Five Colour Theorem

Five colours are sufficient to colour any planar graph. Heawood proved this using the concept of Kempe chain introduced by Alfred Kempe.

Definition 1. *(Kempe chain) An $A-B$* **Kempe chain** *in a properly coloured graph G is a maximal connected subgraph of G with vertices coloured A or B.*

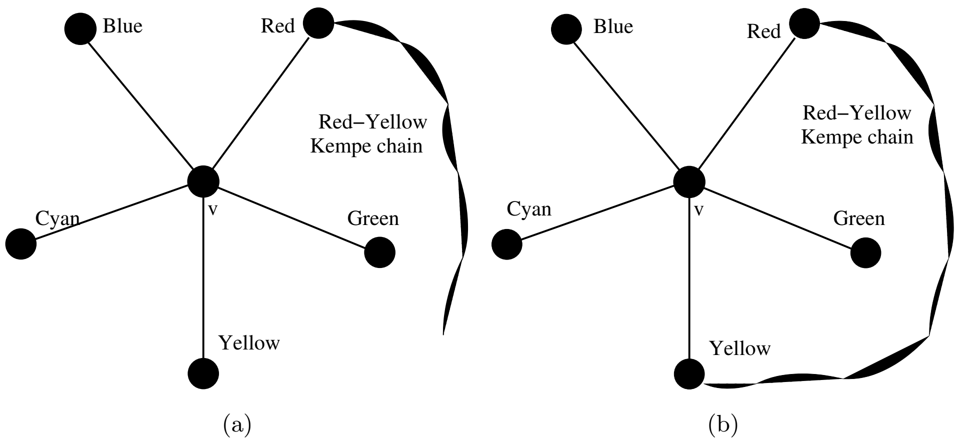


Figure 5

Theorem 6. *(Five Colour Theorem, [2]) Every planar graph is five colourable.*

Proof. The proof is by induction on the number of vertices. Proof is trivial for planar graphs with atmost 5 vertices. Let a planar graph G has $n \geq\ 5$ vertices. Let the vertex with smallest degree be v, where $deg(v) \leq\ 5$. Consider a proper colouring of $G \setminus v$. The difficult case of the proof is when $deg(v) = 5$ and all its neighbours got different colours.

Without loss of generality assume that the neighbours of v are coloured Red, green, Yellow, Cyan and Blue. Consider the Red-Yellow Kempe chain containing the Red coloured neighbour. If this chain doesn't contain the Yellow coloured neighbour (Fig. 5a) then we can toggle the colours (Colour Red vertices with Yellow and vice versa) of the vertices in the chain and colour v with Red. Otherwise assume that the Red-Yellow Kempe chain containing the Red neighbour contains the Yellow neighbour as well (Fig. 5b). In this case there can not be any Green-Blue Kempe chain containing both Green and Blue neighbours of v (why?) . Now, toggle the Green-Blue chain containing Green neighbour and colour v with Green. □

Note that the proof essentially gives us a way to colour a planar graph with 5 colours.

References

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[2] P.J. Heawood. Map-colour theorem. *Proceedings of the London Mathematical Society*, 2(1):161, 1949.

[3] C. L. Liu. *Elements of Discrete Mathematics.* Mc Graw Hill, 1985.

[4] Douglas B. West. *Introduction to Graph Theory.* Prentice Hall of India Private Limited, 2nd edition, 2001.