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Lecture 1: Propositional Calculus

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In this lecture, we will study the logic of propositions. A proposition is a statement which takes value *true* or *false*. We will use propositional variables like p, q, r to denote propositions. Propositional formulas are constructed from variables using the *logical connectives* $\wedge, \vee, \rightarrow$ and $\neg$. Once the *truth values* of the variables of a formula are known, the truth value of the formula can be evaluated. These notions are formalized below.

Syntax of Propositional Calculus

Let V be a collection of propositional variables. The set of Boolean (or propositional) formulas over V denoted by $\mathcal{F}_V$ are inductively defined as follows:

- if $\phi \in V$ then $\phi \in \mathcal{F}_V$.
- if $\phi, \psi \in V$, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$, $(\neg\phi)$, $(\neg\psi)$ are in $\mathcal{F}_V$.

Example 1. *If $V = \{p, q, r\}$ the $(p \wedge (q \rightarrow r))$, $(\neg q \rightarrow (p \vee q))$ etc. are formulas.*

The normal convention is that $\neg$ has the highest precedence among the connectors. $\wedge$ has higher precedence over $\vee$, which in turn has higher precedence over $\rightarrow$ and $\leftrightarrow$. $\wedge$ and $\vee$ are left associative, whereas, $\neg$, $\leftrightarrow$ and $\rightarrow$ are right associative. This allows parenthesis to be omitted. For instance $p \vee q \rightarrow \neg r \wedge p$ denotes $(p \vee q) \rightarrow ((\neg r) \wedge p)$.

Formulas must be given "life" by assigning truth values. This is our next objective. We will use 1 and 0 instead of *true* and *false*.

Semantics of Propositional Calculus

Given a variable set V. A *Truth assignment* for V is a map $\tau : V \longrightarrow \{0, 1\}$ We can extend τ inductively into a function from $\mathcal{F}_V$ to $\{0, 1\}$ (with a little abuse of notation) as follows:

- $\tau(\phi)$ is already defined if $\phi \in V$.
- $\tau(\phi \wedge \psi) = 1$ if both $\tau(\phi) = 1$ and $\tau(\psi) = 1$, 0 otherwise.
- $\tau(\phi \vee \psi) = 1$ if $\tau(\phi) = 1$ or $\tau(\psi) = 1$, 0 otherwise
- $\tau(\phi \rightarrow \psi) = 0$ if $\tau(\phi) = 1$ or $\tau(\psi) = 0$, 1 otherwise
- $\tau(\phi \leftrightarrow \psi) = 1$ if $\tau(\phi) = \tau(\psi)$, 0 otherwise
- $\tau(\neg\phi) = 1$ if $\tau(\phi) = 0$, 1 otherwise.

Example 2. *Let $V = \{p, q, r\}$. Let $\tau(p) = \tau(q) = 1$ and $\tau(r) = 0$. Then $\tau(q \rightarrow r) = 0$, $\tau(p \wedge (q \rightarrow r)) = 0$ $\tau(p \leftrightarrow q) = 1$ etc.*

For the rest of this lecture, we assume that a variable set V is given. The set of all truth assignments from V to $\{0,1\}$ will be denoted by $\mathcal{T}_V$. For $\phi \in \mathcal{F}_V$, we say $\tau \in \mathcal{T}_V$ **satisfies** ϕ if $\tau(\phi) = 1$. The notation $\tau \models \phi$ will be used instead of $\tau(\phi) = 1$. Define $\mathcal{M}(\phi) = \{\tau \in \mathcal{T}_V : \tau \models \phi\}$. The set $\mathcal{M}(\phi)$, called the set of all **models** of ϕ, is the collection of all truth assignments that satisfy ϕ. Let $\phi, \psi \in \mathcal{F}_V$. ψ is said to be a **logical consequence** of ϕ if every $\tau \in \mathcal{M}(\phi)$ satisfies $\tau \models \psi$. That is, whenever a truth assignment makes ϕ true, it should make ψ also true. In such case we write $\phi \Rightarrow \psi$. $\psi \in \mathcal{F}_V$ is said to be **logically equivalent** to $\phi \in \mathcal{F}_V$ if $\mathcal{M}(\phi) = \mathcal{M}(\psi)$. That is, ϕ and ψ are equivalent if every truth assignment in $\mathcal{T}_V$ gives the same truth value to both ψ and ϕ. In this case, we write $\phi \Leftrightarrow \psi$. A formula $\phi \in \mathcal{F}_V$ is said to be **satisfiable** if $\mathcal{M}(\phi) \neq \emptyset$. That is, if there is a truth assignment to the variables that makes the formula evaluate to true. ϕ is said to be **unsatisfiable** otherwise. $\phi \in \mathcal{F}_V$ is a **tautology** if $\mathcal{M}(\phi) = \mathcal{T}_V$. That is, $\tau(\phi) = 1$ for all truth assignments $\tau \in \mathcal{T}_V$. ϕ is **contradictory** if $\mathcal{M}(\phi) = \emptyset$. That is, ϕ evaluates to false under any truth assignment. Note that ϕ is a tautology if and only if $\neg\phi$ is contradictory.

Example 3. *Let $V = \{p, q\}$. The formula $\phi = (p \to q) \to q$ is satisfiable as $\mathcal{M}(\phi) = \{01, 10, 11\} \neq \emptyset$. The formula is not a tautology. Let $\psi = p \vee q$. Then, clearly $\phi \Leftrightarrow \psi$.*

Next we extend these notions to collections of formulas. Let $\mathcal{A} \subseteq \mathcal{F}_V$. Define $\mathcal{M}(\mathcal{A}) = \bigcap_{\phi \in \mathcal{A}} \mathcal{M}(\phi)$. Thus models of $\mathcal{A}$ are precisely those truth assignments that satisfy every formula in $\mathcal{A}$. For each $\tau \in \mathcal{M}(\mathcal{A})$, we write $\tau \models \mathcal{A}$. We set $\mathcal{M}(\emptyset) = \mathcal{T}_V$. $\mathcal{A}$ is **satisfiable** or **consistent** if $\mathcal{M}(\mathcal{A}) \neq \emptyset$. $\mathcal{A}$ is said to be **inconsistent** if $\mathcal{A}$ is not consistent.

Definition 1. *Let $\mathcal{A} \subseteq \mathcal{F}_V$.*

- *$\mathcal{A} \subseteq \mathcal{F}_V$ is said to be* **categorical** *(or sometimes called* **complete***) if $|\mathcal{M}(\mathcal{A})| \leq 1$. That is, either $\mathcal{A}$ is inconsistent or there is a unique $\tau \in \mathcal{T}_V$ such that $\tau \models \mathcal{A}$.*
- *$\phi \in \mathcal{F}_V$ is said to be* **independent** *of $\mathcal{A}$ if both $\mathcal{A} \cup \{\phi\}$ and $\mathcal{A} \cup \{\neg\phi\}$ are consistent. That is, there exists truth assignments $\tau_1, \tau_2 \in \mathcal{M}(\mathcal{A})$ such that $\tau_1 \models \phi$ and $\tau_2 \models \neg\phi$.*
- *$\phi \in \mathcal{F}_V$ is said to be a* **logical consequence** *of $\mathcal{A} \subseteq \mathcal{F}_V$ if every $\tau \in \mathcal{M}(\mathcal{A})$ satisfies $\tau \models \psi$. In this case we write $\mathcal{A} \models \psi$.*
- *$\mathcal{A}, \mathcal{A}' \subseteq \mathcal{F}_V$ are said to be* **logically equivalent** *if $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{A}')$. That is, the set of truth assignments (models) that satisfy all formulas in $\mathcal{A}$ and $\mathcal{A}'$ are exactly the same.*

Note that the sets $\mathcal{A}$ and $\mathcal{A}'$ in these definitions could contain infinitely many formulas from $\mathcal{F}$.

Example 4. *Let $V = \{p, q, r\}$ and $\mathcal{A} = \{p \to q, q \to r, \neg r \vee \neg q\, q \vee r\}$. The set $\mathcal{A}$ is consistent as $\tau(p) = \tau(q) = 0, \tau(r) = 1$ satisfies $\mathcal{A}$. The set is categorical as no other truth assignment satisfies the set. $\neg p$, is an example of a logical consequence of $\mathcal{A}$. Since the set is categorical, there is no formula in $\mathcal{F}_V$ that is independent of $\mathcal{A}$ (why?).*

Example 5. *The set $\mathcal{A} = \{p_1 \vee p_2, p_2 \vee p_3, p_3 \vee p_4, ...\}$ over $V = \{p_1, p_2, ...\}$ is consistent. $\tau(p_i) = 1$ for all i satisfies $\mathcal{A}$. $\mathcal{A}$ is not categorical (why?). $\neg p_2 \to (p_1 \vee p_3)$ is a logical consequence of $\mathcal{A}$ (why?). For each i, the formula $p_i \in \mathcal{F}_V$ is independent of $\mathcal{A}$ (why?).*

Exercise 1. *Show that if $\mathcal{A} \subseteq \mathcal{F}_V$ is categorical, then for every $\phi \in \mathcal{F}_V$ either $\mathcal{A} \cup \{\phi\}$ is inconsistent or $\mathcal{A} \cup \{\neg\phi\}$ is inconsistent. Hence there is no $\phi \in \mathcal{F}_V$ that is independent of a complete set $\mathcal{A} \subseteq \mathcal{F}_V$. Conversely, if there exists $\phi \in \mathcal{F}_V$ independent of $\mathcal{A}$, then show that $\mathcal{A}$ is not categorical.*

Exercise 2. *Let $V = \{p, q, r\}$ Give an example for a consistent and complete set $\mathcal{A} \subseteq \mathcal{F}_V$ and formula $\phi \in \mathcal{F}_V$ such that both $\mathcal{A} \cup \{\phi\}$ and $\mathcal{A} \cup \{\neg\phi\}$ are inconsistent.*

Exercise 3. *Let $\mathcal{A} = \emptyset$. Then a formula $\phi \in \mathcal{F}_V$ is independent of $\mathcal{A}$ if and only if neither ϕ nor $\neg\phi$ are tautologies.*

Definition 2. *Let $\phi, \psi \in \mathcal{F}_V$.*

- ϕ **tautologically implies** ψ *if the formula $\phi \to \psi$ is a tautology. In this case, we write $\phi \Rightarrow \psi$.*
- ϕ *is* **tautologically equivalent** *to ψ if $\phi \leftrightarrow \psi$ is a tautology. In this case we write $\phi \Leftrightarrow \psi$.*

Exercise 4. *Show that a ψ is a logical consequence of ϕ if and only if ϕ tautologically implies ψ.*

Exercise 5. *Show that a ψ is logically equivalent to ϕ if and only if $\phi \leftrightarrow \psi$ is a tautology (that is ϕ is tautologically equivalent to ψ).*

The notions of tautological implication and logical consequence mean exactly the same concept in view of the exercises above. This notion has central importance in *deductions* which we will see as we proceed further through these notes.

The next definition states that a set of formulas form an independent set if there are no *redundant* formulas in the set in the sense none of the formulas in the set is a logical consequence of the others.

Definition 3. *A set $\mathcal{A} \subseteq \mathcal{F}_V$ is said to be an* **independent set of formulas** *if for each $\phi \in \mathcal{A}$, ϕ is independent of $A - \{\phi\}$.*

Example 6. *The set $\mathcal{A} = \{p, p \to q, q \to r, r\}$ over $V = \{p, q, r\}$ is* **not** *an independent set because r is a logical consequence of the remaining formulas.*

Exercise 6. *Let $V = \{p, q, r\}$. In each case determine whether the given axiom set $\mathcal{A}$ is consistent or complete. Whenever $\mathcal{A}$ is incomplete, find a formula $\phi \in \mathcal{F}_V$ that is independent of $\mathcal{A}$.*

1. $\mathcal{A} = \{p \to (q \to r), \neg q, p\}$.
2. $\mathcal{A} = \{p \to q, q \to r, r \to q, \neg q\}$
3. $\mathcal{A} = \{p \vee q, q \wedge r, \neg r\}$.