Department of Computer Science and Engineering, NIT Calicut

Lecture 1: Sets, Functions and Relations

K. Murali Krishnan

This lecture assumes that the reader has some familiarity with sets, relations and functions. We begin with a review of basic definitions primarily to set up the notation.

Let S and T be *non-empty* sets. Let $f : S \longrightarrow T$ be a map (function) from S to T. We called S the *domain* and T the *co-domain* of the function. Let A be a subset of S. We define $f(A) = \{y | y = f(a) \text{ for some } a \in A\}$. In other words $f(A)$ is the *image* of the set A under f. We will succinctly write $f(A) = \bigcup_{a \in A} f(a)$ (instead of $f(A) = \bigcup_{a \in A}\{f(a)\}$). The set $f(S)$ (corresponding to $A = S$) is simply called the *image of f* (instead of image of S under f).

Definition 1. *f is said to be* **injective** *if for every $a, b \in S$, $f(a) \neq f(b)$ unless $a = b$. f is* **surjective** *if $f(S) = T$. A* **bijective** *function is one which is both injective and surjective.*

If $x \in T$, define $f^{-1}(x) = \{a | a \in S, f(a) = x\}$. For $X \subseteq T$, we define $f^{-1}(X) = \bigcup_{x \in X} f^{-1}(x)$. Thus $f^{-1}(X)$ is the collection of all elements in S whose image falls in the set X. Note that if f is surjective, then $f^{-1}(x)$ is non-empty for each $x \in T$. If f is injective, then $f^{-1}(x)$ has and has *at most* one element for each $x \in T$ (why?). In any case, if $x, y \in T$ satisfy $x \neq y$, then $f^{-1}(x) \cap f^{-1}(y) = \emptyset$ (why?). It follows that if X, Y are disjoint subsets of T, $f^{-1}(X)$ and $f^{-1}(Y)$ are disjoint.

Example 1. *Let $S = \{0, 1, 2, 3..\} = T$ and $f(x) = 2x$. Let $A = \{0, 2, 4, 6, ..\}$. Then $f(A) = \{0, 4, 8, ...\}$. It is easy to see that f is injective (prove!) but not surjective. The image of f is $\{0, 2, 4, 6, ...\}$. For $X = \{1, 3, 5, ...\}$, $f^{-1}(X) = \emptyset$.*

Exercise 1. *Let $f : S \longrightarrow T$ be a function. Let A, B be subsets of S and X, Y be subsets of T. We will use the notation A^c, X^c to denote the sets $S - A$, $T - X$ etc. (complements in the respective domains). We prove some, and leave to you the rest.*

1. *$f(A) - f(B) \subseteq f(A - B)$. Give an example to show that equality need not hold true in general. Show that equality holds if f is injective.*

   Proof. If $f(A) - f(B)$ is empty, the claim holds trivially. Otherwise, let $x \in f(A) - f(B)$. As $x \in f(A)$ and $x \notin f(B)$, there exists $a \in A$ such that $a \notin B$ and $f(a) = x$. In other words, there exists $a \in A - B$ such that $f(a) = x$. Thus $x \in f(A - B)$.

   Now, suppose that $f(A - B) \neq \emptyset$ and let $y \in f(A - B)$, there must be some $a \in A - B$ such that $f(a) = y$. Thus $a \in A$ and $a \notin B$ satisfy $y = f(a)$. Clearly, we have $y \in f(A)$. If f is injective, then there cannot be any other $b \neq a$ such that $f(b) = y$. Thus, if f is injective, $y \notin f(B)$. Consequently $y \in f(A) - f(B)$ when f is injective.

   The following simple counterexample shows that equality may fail to hold when f is not injective. Let $S = \{a, b\}, T = \{x\}$. Let $f(a) = f(b) = x$. Let $A = \{a\}$ and $B = \{b\}$. Thus $A - B = \{a\}$ and $f(A - B) = \{x\}$. However, $f(A) = f(B) = \{x\}$ and hence $f(A) - f(B) = \emptyset$. □

2. *$f(A \cup B) = f(A) \cup f(B)$.*

3. *$f(A \cap B) \subseteq f(A) \cap f(B)$. Give an example to show that the inequality is strict. Show that equality holds if f is injective.*

4. *$A \subseteq f^{-1}(f(A))$, equality holds when f is injective.*

5. *$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$.*

6. *$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.*

   Proof. First we prove that $f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$. If $f^{-1}(X \cap Y) = \emptyset$, the claim holds trivially. Otherwise, Let $a \in f^{-1}(X \cap Y)$. Then, there exists $z \in X \cap Y$ such that $z = f(a)$. As $z \in X$ and $z \in Y$, $a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$. Consequently, $a \in f^{-1}(X) \cap f^{-1}(Y)$.

   Conversely, we prove that $f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X \cap Y)$. Let $a \in f^{-1}(X) \cap f^{-1}(Y)$. Thus $a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$. Consequently, There exists $x \in X$ such that $f(a) = x$ and there exists $y \in Y$ such that $f(a) = y$. However, since f is a function, we must have $x = y$. Consequently, there exists $x \in X \cap Y$ such that $f(a) = x$. That is, $a \in f^{-1}(X \cap Y)$. □

7. *$f^{-1}(X - Y) = f^{-1}(X) - f^{-1}(Y)$*

8. *If f is surjective, then $[f(A)]^c \subseteq f(A^c)$. (What goes wrong if f is not surjective?).*

9. *$f^{-1}(X^c) = [f^{-1}(X)]^c$.*

10. *$f(f^{-1}(X)) = X$.*

If $R \subseteq S \times T$ is a relation, and $a \in S$, define $R(a) = \{y : (a, y) \in R\}$. Thus $R(a)$ denotes the set of elements in R which are "related to" a. If $A \subseteq S$, we define $R(A) = \bigcup_{a \in A} R(a)$. With this notation, a relation R from S to T can be viewed as a function $f_R : S \mapsto 2^T$ (where 2^T is the power set of T) by defining $f_R(a) = R(a)$. This allows relations to be viewed as functions. Conversely, a function $f : S \mapsto T$ can be thought of as a relation $R_f - \{(a, f(a)) : a \in S\}$. Observe that f^{-1} is function if and only if f is bijective (why?).

Just as with functions, for $x \in T$, we define $R^{-1}(x) = \{a | a \in A \text{ and } (a, x) \in R\}$. For $X \subseteq T$, define $R^{-1}(X) = \bigcup_{x \in X} R^{-1}(x)$.

Exercise 2. *Let $R \subseteq S \times T$ be a relation. Let A, B be subsets of S and X, Y be subsets of T. Let A^c, X^c denote the sets $S - A$, $T - X$ etc. In each of the following cases, check whether the set on the left/right is included in the set of right/left. (Hint: Some results follow easily if you note that a relation from S to T is a function from S to 2^T. In some cases you may be able to find examples to show that neither side is a subset of the other always).*

1. *$R(A) - R(B)$ and $R(A - B)$.*

2. *$R(A \cup B)$ and $R(A) \cup R(B)$.*

3. $R(A \cap B)$ *and* $R(A) \cap R(B)$.

4. A *and* $R^{-1}(R(A))$.

5. $R^{-1}(X \cup Y)$ *and* $R^{-1}(X) \cup R^{-1}(Y)$.

6. $R^{-1}(X \cap Y)$ *and* $R^{-1}(X) \cap R^{-1}(Y)$.

7. $R^{-1}(X - Y)$ *and* $R^{-1}(X) - R^{-1}(Y)$

8. $[R(A)]^c$ *and* $R(A)^c$.

9. $R^{-1}(X^c) = [R^{-1}(X)]^c$.

10. $R(R^{-1}(X))$ *and* X.

Exercise 3. *Let S be the set of real numbers. Consider the relation $R \subseteq S \times S$ given by $R = \{(x, y) : x + y = 1\}$. Let X be the set of all integers. Let Y be the set of positive real numbers. Find 1.) $R(X)$, 2.) $R(Y)$, 3.) $R^{-1}(X)$, 4.) $R^{-1}(Y)$. 5.) $R(R^{-1}(Y)$ 6.) $R^{-1}(R(Y))$*

Exercise 4. *Let S be the set of positive integers. Consider the relation $R \subseteq S \times S$ given by $R = \{(a, b) : GCD(a, b) = 1\}$. Let X be the set of all prime numbers and let Y be the set of all even numbers. Find 1.) $R(X)$, 2.) $R(Y)$, 3.) $R^{-1}(X)$, 4.) $R^{-1}(Y)$. 5.) $R(R^{-1}(Y)$ 6.) $R^{-1}(R(Y))$*

Definition 2. *Let R be a relation over a set S. define the* **complement** *of R, $\overline{R} \subseteq S \times T$ as $R = \{(a, b) : (a, b) \notin R\}$.*

Exercise 5. *Let S, T be non-empty sets. Let $R_1, R_2 \subseteq S \times T$. Show that*

1. $\overline{R_1 \cup R_2} = \overline{R_1} \cap \overline{R_2}$, $\overline{R_1 \cap R_2} = \overline{R_1} \cup \overline{R_2}$.

2. $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$, $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$.