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Lecture 2: Deduction in Propositional Logic

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In modeling a real system, we start with a primitive set of variables V and postulate a set $\mathcal{A}$ of defining properties of the system as the *axioms* or *postulates*. The axioms must be adequate so that all true formulas about the the system (*theorems*) are logical consequences of the axioms. A desirable property of an axiom system is to have an independent set of axioms, so that the axioms contain no redundancies.

Given an axiom system $\mathcal{A}$ and a formula ϕ, determining whether $\mathcal{A} \models \phi$ is called the *deduction problem*. Attempting all truth assignments brute force is impractical. Instead, we try to identify a formula ψ already known to satisfy $\mathcal{A} \models \psi$ (for example ψ could be an axiom) and show that $\psi \Rightarrow \phi$. It will follow that $\mathcal{A} \models \phi$. This essentially is the fundamental notion of logical *deduction* used in mathematical reasoning.

To execute this plan, one needs a good database of "standard" tautological implications. Some of them are developed in the following exercises. The following section formalizes the notions of axiomatic systems, deduction, theorem-hood etc.

Example 1. *The formulas* $p \vee \neg p$, $\neq (p \wedge \neg p)$ *(called* **laws of negation***),* $p \wedge (p \to q) \Rightarrow q$ *(***Modus Ponens***),* $(p \to q) \wedge \neg q \Rightarrow \neg p$ *(***Modus Tollens***),* $(p \vee q) \wedge \neg p \Rightarrow q$ *(***disjunctive syllogism***)* $(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$ *(***hypothetical syllogism***),* $(p \to q) \Leftrightarrow (\neg q \to \neg p)$ *(***Law of counter positive***),* $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$, $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ *(De-Morgan's Laws),* $(p \to q) \Leftrightarrow (\neg p \vee q)$, $\neg\neg p \Leftrightarrow p$ *(Law of double negation) etc. are standard examples of tautologies.* $\wedge$, $\vee$ *are* **associative**, **commutative** *and* **distribute** *over each other. The following properties:* $(p \wedge p) \Leftrightarrow (p \vee p) \Leftrightarrow p$, $(p \wedge q) \Rightarrow p$, $(p \wedge q) \Rightarrow q$ *are collectively called the* **absorption properties**. $p \Rightarrow (p \vee q)$ *is called* **generalization**.

As an example, to verify modus ponens, suppose $\tau(p \wedge (p \to q) \to q) = 0$ *Then* $\tau(q) = 0$ *and* $\tau(p \wedge (p \to q)) = 1$. *The latter requires* $\tau(p) = 1$ *and* $\tau((p \to q)) = 1$. *But as* $\tau(q) = 0$, *for* $\tau((p \to q)) = 1$, *we need* $\tau(p) = 0$ *which is a contradiction. (Another standard verification method is using truth tables) Other formulas are verified similarly.*

Another important technique is **substitution**. Suppose we replace a variable with any formula uniformly in a tautology ϕ, then the resultant formula is also a tautology. For instance in the tautology $\phi = p \vee \neg p$, if we substitute p everywhere with $(p \to (q \vee r)$ (written $\phi[p \leftarrow (p \to (q \vee r))]$) we get $(p \to (q \vee r) \vee \neg(p \to (q \vee r))$ which also is a tautology. This is because ϕ evaluates to 1 under any value to the expression. Similarly, since equivalent formulas evaluate to the same truth value, any formula in an expression can be substituted by a logically equivalent formula without affecting truth values. These rules are called **substitution laws** and are summarized below for easy reference.

Theorem 1 (Laws of Substitution).

1. *If* $\phi \in \mathcal{F}_V$ *is a tautology. Let* $p \in V$ *be a variable in* ϕ. *Let* $\psi \in \mathcal{F}_V$. *Then formula* $\phi[p \leftarrow \psi]$ *obtained by substitution of all occurrence of* p *in* ϕ *with* ψ *in* ϕ *also is a tautology.*

2. *Let* $\phi, \alpha.\beta \in \mathcal{F}_V$, *if* α *is a subformula of* ϕ *and* $\alpha \Leftrightarrow \beta$, *Let* ϕ' *be the formula obtained by replacing of* α *with* β *in* ϕ. *Then for every* $\tau \in \mathcal{T}_V$, $\tau(\phi) = \tau(\phi')$.

Classical Deduction

Definition 1. *Let $\mathcal{A} \subseteq \mathcal{F}_V$. Define the set $Th(\mathcal{A}) = \{\phi \in \mathcal{F}_V : \mathcal{A} \models \phi\}$. This set is the collection of all formulas which are logical consequences of $\mathcal{A}$.*

Definition 2. *Let $\tau \in \mathcal{T}_V$. Define $\mathcal{F}(\tau) = \{\phi \in \mathcal{F}_V : \tau \models \phi\}$. $\mathcal{F}(\tau)$ is the collection of all formulas which are satisfied by τ. Let $\mathcal{T} \subseteq \mathcal{T}_V$. Define $\mathcal{F}(\mathcal{T}) = \bigcap_{\tau \in T} \mathcal{F}(\tau)$ to be the collection of all formulas satisfied by every truth assignment in T.*

Axiomatic deduction methods were known right from the time of the ancient Greeks and an axiomatic treatment of geometry can be found in Euclid's book "The Elements". The assumptions about the system under study were postulated as axioms for the system and logical consequences of these axioms were derived using a set of "standard" deduction rules. The derived consequences are called theorems. The notion of algorithmic (automated) deduction did not exist during those times and deductions had to be done manually. In this section, we shall discuss a few deduction methods well known from from the ancient times.

Theorem 2 (Laws of Deduction)**.** *Let $\mathcal{A} \subseteq \mathcal{F}_V$ and $\phi, \psi \in \mathcal{F}_V$. Then:*

- *(Deduction Theorem): $\mathcal{A} \models \phi$ and $(\phi \Rightarrow \psi)$ then $\mathcal{A} \models \psi$.*
- *(Law of Implication): $\mathcal{A} \models (\phi \rightarrow \psi)$ if and only if $\mathcal{A} \cup \{\phi\} \models \psi$*
- *(Method of Contradiction): $\mathcal{A} \cup \{\neg\phi\}$ is inconsistent if and only if $\mathcal{A} \models \phi$.*

Proof. The first statement is proved here and the rest are left as exercises. Suppose $\mathcal{A} \models \phi$. Suppose $\tau \in \mathcal{T}_V$ satisfies $\tau \models \mathcal{A}$. Then, by hypothesis, $\tau \models \phi$. As $\phi \Rightarrow \psi$ is a tautology, $\tau \models \psi$ (why?). This proves the first part. The other parts are proved similarly. □

Example 2. *Let $\mathcal{A} = \{p \rightarrow q, q \rightarrow \neg(r \rightarrow p), \neg r \vee \neg q\}$. Here is a deduction for $\neg p$:*

1. $q \rightarrow \neg(r \rightarrow p)$ *(axiom)*
2. $q \rightarrow \neg(\neg r \vee p)$ *(Substitution: $(r \rightarrow p) \Leftrightarrow (\neg r \vee p)$).*
3. $q \rightarrow (r \wedge \neg p)$ *(Substitution: De-Morgan's equivalence).*
4. $p \rightarrow q$ *(axiom)*
5. $p \rightarrow (r \wedge \neg p)$ *(Hypothetical Syllogism from 4,3)*
6. $(\neg p \vee (r \wedge \neg p)$ *(Substitution)*
7. $(\neg p \vee r) \wedge (\neg p \vee \neg p)$ *(Distributive law).*
8. $\neg p \vee \neg p$ *(Absorption law)*
9. $\neg p$. *(Absorption law)*

Exercise 1. *It is true that Mike has a bike. If Mike has a bike, then Mike can't have a car. If mike does not have a car, then Mike can't travel long distance. Either Mike travels long distance or Mike is a sportsman. Formulate the above statements in propositional logic. Is the statement "Mike is a sportsman" a valid consequence of these statements? If so, find a deduction for the statement based on the above laws. Are these statements consistent? Do they form a categorical set?*

Exercise 2. *Let* $V = \{p, q, r\}$. *Consider* $\mathcal{A} = \{p \to (p \to q), q \to (r \vee p), (\neg p \vee \neg q) \wedge \neg r\}$. *Is* $\mathcal{A}$ *categorical? Is* p *a logical consequence of the axioms? If so, give a deductive proof.*