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Lecture 2: Equivalence Relations and Partitions

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Let S be a set. Let $R \subseteq S \times S$. We say that R **is relation defined over** S. We will write aRb as a short hand notation for $(a, b) \in R$ (or $b \in R(a)$).

Definition 1. *A relation R defined over S is said to be* **reflexive** *if aRa for all $a \in S$. R is* **symmetric** *if whenever aRb holds, bRa also holds for all $a, b \in S$. R is said to be* **transitive** *if for all $a, b, c \in S$, whenever aRb and bRc, we have aRc. R is said to be* irreflexive, asymmetric or intransitive *when it is respectively not reflexive or not symmetric or not transitive. R is said to be* **anti-symmetric** *if both aRb and bRa will hold simultaneously if and only if $a = b$ for any $a, b \in S$.*

Note that the *equality relation* $' =' = \{(a, a)|a \in S\}$ is reflexive, symmetric and transitive (why?).

The notation **C**, **R**, **Q**, **Z** and **N** respectively will denote the set of complex, real, rational, integer and natural numbers. We use $\mathbf{Z_n}$ to denote the set $\{0, 1, 2, .., n - 1\}$. Addition and multiplication in $\mathbf{Z_n}$ is performed $\mod n$. Thus, if $a, b \in \mathbf{Z_n}$, then $a + b$ denotes $a + b \mod n$ and ab denotes $ab \mod n$. The '=' symbol will always represent the identify relation (that is, $\{(a, a) : a \in S\}$, where S is the domain S under consideration). A relation R is a function when for each $a \in S$, $R(a)$ is a singleton set.

Exercise 1. *Consider the following examples:*

1. *Let $S = \mathbf{R} \times \mathbf{R}$ (the set of pairs of reals or simply the Cartesian plane). Let $R = \{(x, y)|x + y = 1\}$. Is R a function? If so is it surjective? injective? bijective? If bijective, find f^{-1}.*

2. *Let $S = \mathbf{N} = \{0, 1, 2...\}$. Let $R = \{(a, b)|a \text{divides } b\}$. (Note 0 does not divide any number, but every non-zero element in $\mathbf{N}$ divides 0). Is R reflexive? symmetric? anti-symmetric? transitive?*

3. *Let $S = \mathbf{Z} \times \mathbf{N} - \{0\}$. (That is S consists of pairs of integers, with the second number in the pair being a positive integer). Let $R = \{((p, q), (r, s))|ps = qr\}$. Is R a) reflexive? b) symmetric? c) anti-symmetric? d) transitive?*

4. *Show that if R (over S) is reflexive, anti-symmetric, symmetric and transitive, then R must be the identity relation.*

5. *Let $S = \mathbf{R} \times \mathbf{R}$, show that the relation $R = \{((x, y), (x'y'))|ax + by = ax' + by'\}$ is reflexive, symmetric and transitive. What is the geometric property captured by the relation? (That is, if points (x, y) and $(x'y')$ are related, what can you say about their positions in the two dimensional coordinate system?)*

6. *Show that a relation R defined over any set S is symmetric if and only if $R = R^{-1}$.*

Definition 2. *A relation is an* **equivalence relation** *if it is reflexive, symmetric and transitive.*

The notation $M_n(\mathbf{R})$ or $M_n(\mathbf{C})$ will denote $n \times n$ matrices with real/complex entries. $GL_n(\mathbf{R})$ or $GL_n(\mathbf{C})$ denotes the set of $n \times n$ non-singular real matrices (i.e., $n \times n$ real matrices with non-zero determinant).

If n is a positive integer and a, b integers, the notation $a \equiv b \mod n$ will be used to say that $(a - b)$ is divisible by n.

Exercise 2. *Show that the relation R defined in each case is an equivalence relation.*

1. *In $M_n(\mathbf{R})$ define the relation $R = \{(A, B) : \exists P \in GL_n(\mathbf{R}) such\ that A = PBP^{-1}\}$. We say matrices A and B are* **similar** *if $(A, B) \in R$. Find is $R^{-1}(GL_n(\mathbf{R}))$?*

2. *Let n be a positive integer. Over $\mathbf{Z}$, define $R = \{(a, b) : a \equiv b \mod n\}$.*

Exercise 3. *If R_1, R_2 are equivalence relations over a set S, so is $R_1 \cap R_2$. Is it always true that $R_1 \cup R_2$ is an equivalence relation?*

A collection of *disjoint* subsets of a set S define a *partitioning* of S if each set in the collection is non-empty and their union is S.

Definition 3. *Let S be any set. A collection $\{S_i\}_{i \in I}$ of subsets of S is a* **partitioning** *of S if 1.) $S_i \neq \emptyset$ for each $i \in I$ 2.) $S_i \cap S_j = \emptyset$ if $i \neq j$ and 3. $\bigcup_{i \in I} S_i = S$. Each set S_i in a given partitioning of S is called an* **equivalence class** *in the partitioning.*

We next investigate the connection between equivalence relations and partitions.

Example 1. *The following are examples of partitions.*

1. *Let S, T be non-empty sets. Let $f : S \mapsto T$ be a* surjective *function. Then, $\{f^{-1}(t)\}_{t \in T}$ is a partitioning of S.*

2. *Consider $S = \mathbf{R}^2$. Let a, b be positive real numbers. For each real number t, define $S_t = \{(x, y) : x, y \in \mathbf{R}, ax + by = t\}$. Show that $\{S_t\}_{t \in \mathbf{R}}$ is a partitioning of S. (Each equivalence class consists of points in a line parallel to $ax + by = 0$.)*

3. *Consider $S = \mathbf{Z}$. Let n be a positive integer. Let $I = \in \{0, 1, 2, \ldots, n - 1\}$ For each $k \in I$ define $S_k = \{j \in \mathbf{Z} : j \equiv k \mod n\}$. Show that $\{S_k\}_{k \in I}$ is a partitioning of $\mathbf{Z}$.*

Exercise 4. *Let S be a non-empty set and let I be any non-empty index set. Let $\{S_i\}_{i \in I}$ be a partitioning of S. Define relation R on S, $R = \{(a, b) : \exists i \in I, a, b \in S_i\}$. Show that R is an equivalence relation.*

The next exercise shows that every equivalence relation R defined on a set S induces a partitioning of S. Hence, in view of the exercise above, the notions of equivalence relations and partitions are the same.

Exercise 5. *Let R be an equivalence relation defined on a non-empty set S. This exercise proves that $\{R(a)\}_{a \in S}$ is a partitioning of S.*

1. *Show that for each $a \in S$, $R(a) \neq \emptyset$.*
2. *Show that $\bigcup_{a \in S} R(a) = S$.*
3. *Show that for any $a, b \in S$, $a \neq b$, $R(a) \cap R(b) \neq \emptyset$ the $R(a) = R(b)$.*

   Proof. Suppose $a \neq b$ and $R(a) \cap R(b) \neq \emptyset$. Let $z \in R(a) \cap R(b)$. Let x be an arbitrary element in $R(a)$. We will show that $x \in R(b)$ as well, proving that $R(a) \subseteq R(b)$. The proof $R(b) \subseteq R(a)$ is similar.

   By assumption aRz. By symmetry of R we conclude zRa. Similarly, by assumption bRz. From transitivity of R, we conclude bRa. Now, by symmetry of R we get aRb

   Since $x \in R(a)$, we have aRx. By symmetry we get xRa. From xRa and aRb, by transitivity of R, we get xRb. By symmetry, we have bRx or $x \in R(b)$. □

Exercise 6. *How many equivalence relations are possible over the set $\{1, 2, 3\}$?. How may equivalence relations are possible over the set $S = \{1, 2, 3, 4\}$?*

Definition 4. *Let R be an equivalence relation over a non-empty set S. The* **index** *of R is the number of equivalence classes in the partition of S $\{R(a)\}_{a \in S}$ defined by R.*

Exercise 7. *Let $S = \mathbf{R}$. Let R be defined by xRy if $x - y \in \mathbf{Z}$. Show that R is an equivalence relation. Show the the index of R is not finite.*

Exercise 8. *Let $n = 24$ and $S = \mathbf{Z}_n$. Define $R = \{(a, b) \in \mathbf{Z}_n : GCD(a, n) = GCD(b, n)\}$. Show that R is an equivalence relation. What is the index of R?*

Exercise 9. *Let S be any non-empty set. Consider the set $2^{S \times S}$ (that is collection of all subsets of of $S \times S$) be the set of all relations defined over S. Define an injective function $f : S \mapsto 2^{S \times S}$ such that: 1.) f is injective 2.) For each $a \in S$, $f(a)$ is an equivalence relation. (This exercise shows that the number of equivalence relations over a non-empty set is "as least as many" as the number of elements in S).*