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Lecture 3: First Order Logic for Graphs

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In this lecture, we will study first order logic for graphs (with equality) denoted by $FOLG(=)$. A graph $G = (V, E)$ consists of a finite or countably infinite set of vertices and a collection of (directed) edges $E \subseteq V \times V$. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** (written $G_1 \cong G_2$) if there is a bijective map $f : V_1 \longrightarrow V_2$ such that $(v, v') \in E_1$ if and only if $(f(v), f(v')) \in E_2$. For instance, $G_1 = (\{1, 2\}, \{(1, 2), (2, 2)\})$ and $G_2 = (\{a, b\}, \{(a, b), (b, b)\})$ are isomorphic via the map $f(1) = a, f(2) = b$ from V_1 to V_2. Isomorphic graphs are essentially copies of the same graph with a different labelling of the vertices. We do not distinguish between isomorphic graphs and treat them as the same graph.

Syntax of $FOLG(=)$

The vocabulary of $FOLG(=)$ consists of **variables** $X = \{x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, ...\}$, **logical operators** $\{\wedge, \vee, \neg, \rightarrow, \leftrightarrow\}$, the **quantifiers** $\{\forall, \exists\}$ and the two binary relations relations $\{R, =\}$. The set $\mathcal{F}$ of **formulas** in $FOLG(=)$ are defined as follows:

- $R(x, y)$ and $(x = y)$ are in $\mathcal{F}$ whenever $x, y \in X$.
- If $\phi, \psi \in V$, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$, $(\neg\phi)$, $(\neg\psi)$ are in $\mathcal{F}$.
- If $x \in X$ and $\phi \in \mathcal{F}$, then $(\forall x\phi)$, $(\forall x\psi)$, $(\exists x\phi)$, $(\exists x\psi)$ are in $\mathcal{F}$.

Example 1. $\forall x(R(x, y) \rightarrow \neg R(y, x))$, $\exists x(\forall y(R(x, y) \rightarrow (x = y)) \wedge (\exists y \neg R(x, y)))$ *etc. are syntactically correct formulas. In the second formula, note that the whole formula is within the* **scope** *of the existential quantifier* $\exists x$ *whereas each of the two* subformulas *contain the variable* y *which is under the scope of different quantifiers. Normal rules of parenthesizing and scope resolution applies and we do not formally illustrate the scope rules here. The variable* y *appearing in the first formula is not under the scope of any quantifier and is called a* **free variable**. *A formula is said to be* **closed** *if it does not contain any free variable. .*

Semantics of $FOLG(=)$

Formulas come to life and gets true or false values when variables in X and the relation R are *interpreted* over a graph. Let $G = (V, E)$ be a graph. Define an **assignment** $\tau : X \longrightarrow V$ to be any function that maps each variable in X to a vertex in G. Denote by $\mathcal{T}_G$ (written simply $\mathcal{T}$ when the underlying graph is clear from the context) the set of all assignments to X to $V(G)$. The notation $V(G)$ and $E(G)$ will be used to represent the vertex and edge sets of G. We will simply write V and E when the underlying graph is clear.

Let $v \in V$ and $x \in X$ and $\tau \in \mathcal{T}$. We define the new assignment $\tau_{x=v}(y)$, (*substitution of* x *with* v *in* τ) by the rule: $\tau_{x=v}(y) = \tau(y)$ for all $y \in X \setminus \{x\}$ and $\tau_{x=v}(y) = v$ if $y = x$. The function $\tau_{x=v}$ essentially is identical to τ for each variable in X except for the variable x which is re-assigned the new value v. If x, y are distinct variables, it is not hard to see that order in which substitutions are done to x and y does not matter, and hence we write $\tau_{x=u,y=v}$ for the composition of the substitutions $\tau_{x=u}$ and $\tau_{y=v}$.

Let $\phi \in \mathcal{F}$. The notation $(G, \tau) \models \phi$ will mean that the formula ϕ is true in the graph G when the variables in X are assigned values according to τ (read as G with assignment τ satisfies ϕ). Here is the formal definition:

Then for any $x, y, z \in X$, Define:

- $(G, \tau) \models R(x, y)$ if $(\tau(x), \tau(y)) \in E$, $(G, \tau) \not\models R(x, y)$ otherwise.
- $(G, \tau) \models (x = y)$ if $\tau(x) = \tau(y)$, $(G, \tau) \not\models (x = y)$ otherwise.
- $(G, \tau) \models (\phi \vee \psi)$ if $(G, \tau) \models \phi$ or $(G, \tau) \models \psi$, $(G, \tau) \not\models (\phi \vee \psi)$ otherwise.
- $(G, \tau) \models \neg\phi$ if $(G, \tau) \not\models \phi$, $(G, \tau) \not\models \neg\phi$ otherwise.
- $(G, \tau) \models (\forall x \phi)$ if for each $v \in V$, $(G, \tau_{x=v}) \models \phi$, $(G, \tau) \not\models (\forall x \phi)$ otherwise.
- $(G, \tau) \models (\exists x \phi)$ if for at least one $v \in V$, $(G, \tau_{x=v}) \models \phi$, $(G, \tau) \not\models (\forall x \phi)$ otherwise.

The remaining connectives ($\wedge$ and $\rightarrow$) can be derived from the above.

Exercise 1. *Let* $\phi \in \mathcal{F}$. *Show that:*

- $(G, \tau) \models \forall x \phi$ *if and only if* $(G, \tau) \models \neg\exists x \neg\phi$.
- $(G, \tau) \models \exists x \phi$ *if and only if* $(G, \tau) \models \neg\forall x \neg\phi$.

A careful reflexion on the definition of satisfiability leads to the observation that the truth of a formula involving only quantified variables does not depend any particular assignment of values to the variables:

Lemma 1. *Let* $\phi \in \mathcal{F}$ *is closed and let* $G = (V, E)$ *be a graph, then* $(G, \tau) \models \phi$ *for some* $\tau : X \longrightarrow V$ *if and only if* $(G, \tau) \models \phi$ *for every* $\tau : X \longrightarrow V$. *Hence when* ϕ *is closed, we simply write* $G \models \phi$ *or* $G \not\models \phi$ *without referring to any assignment (and use descriptions like* G *satisfies/models* ϕ *or* G *does not satisfy/model* ϕ*).*

The notions of of satisfiability, consistency, categoricalness, model, logical consequence etc. in first order logic mirror the equivalent concepts in propositional logic.

Definition 1. *For a given graph* G*, the notation* τ, τ' *etc. will be used to denote various assignments to variables in* X *with values in* $V(G)$. ϕ, ψ *etc. will denote formulas in* $\mathcal{F}$. **We will assume here that all formulas are closed**

- *Define* $\mathcal{M}_\phi = \{G : G \models \phi\}$. *This is the collection of models for* ϕ. *A closed formula* ϕ *is said to be* **satisfiable or consistent** *if* $\mathcal{M}_\phi \neq \emptyset$. *Thus* ϕ *is satisfiable if it has at least one model.*

- *For $\mathcal{A} \subseteq \mathcal{F}_V$, $\mathcal{M}(\mathcal{A}) = \bigcap_{\phi \in \mathcal{A}} \mathcal{M}_\phi$. This is the collection of all graphs that are satisfies every formula in $\mathcal{A}$. $\mathcal{A}$ is said to be* **satisfiable or consistent** *if $\mathcal{M}(\mathcal{A}) \neq \emptyset$. The notation $G \models \mathcal{A}$ will be sometimes used for denoting $G \in \mathcal{M}(\mathcal{A})$.*

- *$\mathcal{A} \subseteq \mathcal{F}$ is said to be* **categorical** *if whenever $G, G' \in \mathcal{M}(\mathcal{A})$, G is isomorphic to G'. That is, either $\mathcal{A}$ is inconsistent or there is a unique graph G (upto isomorphism) that satisfies $\mathcal{A}$.*

- *$\phi \in \mathcal{F}$ is said to be* **independent** *of $\mathcal{A} \subseteq \mathcal{F}$ if both $\mathcal{A} \cup \{\phi\}$ and $\mathcal{A} \cup \{\neg\phi\}$ are consistent. That is, there exists graphs G_1, G_2 such that $G_1 \models \mathcal{A} \cup \{\phi\}$ and $G_2 \models \mathcal{A} \cup \{\neg\phi\}$*

- *$\psi \in \mathcal{F}$ is said to be a* **logical consequence** *of ϕ if every $G \in \mathcal{M}_\phi$ also satisfies $G \models \psi$. In this case we write $\phi \Rightarrow \psi$.*

- *$\psi \in \mathcal{F}$ is said to be* **logically equivalent** *to $\phi \in \mathcal{F}$ if for every $G \in \mathcal{G}$, $G \models \phi$ if and only if $G \models \psi$. That is, $\mathcal{M}_\phi = \mathcal{M}_\psi$. In this case, we write $\phi \Leftrightarrow \psi$.*

- *$\phi \in \mathcal{F}$ is said to be a* **logical consequence** *of $\mathcal{A} \subseteq \mathcal{F}$ if every $G \in \mathcal{M}(\mathcal{A})$ satisfies $G \models \psi$. In this case we write $\mathcal{A} \models \psi$.*

- *$\mathcal{A}, \mathcal{A}' \subseteq \mathcal{F}$ are said to be logically equivalent if $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{A}')$. That is, the set of models for $\mathcal{A}$ and $\mathcal{A}'$ are precisely the same.*

- *$\phi \in \mathcal{F}$ is a* **tautology** *if $\mathcal{M}_\phi$ contains every graph.*

- *ϕ is* **contradictory** *if $\mathcal{M}_\phi = \emptyset$. That is ϕ is always false. Note that ϕ is a tautology if and only if $\neg\phi$ is contradictory.*

Note that the sets $\mathcal{A}$ and $\mathcal{A}'$ in these definitions could contain infinitely many formulas from $\mathcal{F}$.

Example 2. *Let $\mathcal{A} = \{\forall x R(x,x), \forall x \forall y [R(x,y) \rightarrow R(y,x)], \forall x \forall y \forall z [R(x,y) \wedge R(y,z) \rightarrow R(y,z)]\}$. $\mathcal{M}(\mathcal{A})$ consists of the graphs corresponding to equivalence relations. If the second formula is replaced with $\forall x \forall y (R(x,y) \wedge R(y,x) \rightarrow (x = y))$, the models are the collection of all graphs of partially ordered sets. Note that these axioms are consistent and non-categorical (why?).*

Exercise 2. *Write down an axiom set $\mathcal{A}$ whose models are lattices.*

Exercise 3. *Let $\mathcal{A} = \{\forall x \forall y R(x,y), \exists x \exists y \forall z ((z = x) \vee (z = y))\}$. Find all non-isomorphic graphs that satisfy $\mathcal{A}$. Is $\mathcal{A}$ categorical?*

Exercise 4. *Write down an axiom set $\mathcal{A}$ such that $G \in \mathcal{M}(\mathcal{A})$ if and only if G has infinitely many vertices.*

Exercise 5. *Write down an axiom set $\mathcal{A}$ that describe all graphs that are unions of infinite two sided chains. (A two sided chain is a graph isomorphic to the following graph G with $V = \mathbf{Z}$ and $E = \{(i, i+1) : i \in \mathbf{Z}\}$. (Hint: You will need all axioms of the previous exercise).*