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Lecture 3: Partially Ordered Sets and Lattices

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A relation R defined on a non-empty set S is **antisymmetric** if for all $x, y \in S$, $x = y$ holds whenever xRy, yRx holds. A set S together with a relation R defines a partial order (written as a pair (S, R)) if R is reflexive, antisymmetric and transitive. (S, R) is a **chain** or a **linear order** or a **total order** if additionally R satisfies the property that for all $x, y \in S$, either xRy or yRx. We will use $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$ and $\mathbf{R}$ to denote the set of natural numbers (zero included), integers, rational numbers and real numbers.

Example 1. *The following are examples for partial orders.*

1. *Let $S \in \{\mathbf{N}, \mathbf{Q}, \mathbf{R}, \mathbf{Z}\}$ and $R = \{(a, b) : a \leq b\}$, where $\leq$ denotes the standard ordering of numbers.*

2. *Let $S = \mathbf{N}$. Define $(a, b) \in R$ if $\exists k \in \mathbf{Z}$ such that $b = ak$, the* **divisibility** *relation.*

3. *Let T be any set. Let $S = 2^T$. Define $R = \{(A, B) : A, B \in S, A \subseteq B\}$.*

4. *Let $S = \{0, 1\}^n$, the set of n bit binary vectors. Define $(x_1, x_2, \ldots, x_n)R(y_1, y_2, \ldots, y_n)$ if $x_i \leq y_i$ for each i.*

5. *Let $S = \mathbf{R} \times \mathbf{R}$. Define $(x, y)R(x', y')$ if $x \leq x'$ and $y \leq y'$.*

6. *Let S be the set of all functions from $\mathbf{R}$ to $\mathbf{R}$. For $f, g \in S$, define $(f, g) \in R$ if $f(x) \leq g(x)$ for each $x \in \mathbf{R}$.*

Exercise 1. *Which among the preceding examples are chains?*

We use the notation $\leq$ (instead of R) to denote an arbitrary partial order defined over a non-empty set S. Let $(S, \leq)$ be a partial order. Let $\emptyset \neq A \subseteq S$. The **upper bound** of A is defined by $\mathrm{UB}(A) = \{x \in S : a \leq x \text{ for all } a \in A\}$. Similarly, the **lower bound** of A is defined by $\mathrm{LB}(A) = \{x \in S : x \leq a \text{ for all } a \in A\}$. An element $a_0 \in A$ is a **minimal element** in A if for all $a \in A - \{a_0\}$, $a \nleq a_0$. Thus there is no other element in A ordered strictly below a_0. Similarly, a_0 is a **maximal element** of A if for all $a \in A - \{a_0\}$, $a_0 \nleq a$. A set A may fail to have minimal or maximal elements. $a_0 \in A$ is a **least element (or minimum element)** in A if $a_0 \leq a$ for all $a \in A$. $a_0 \in A$ is the **greatest element (or maximum element)** in A if for all $a \in A$, $a \leq a_0$. The next exercise shows that if a set has a minimum (maximum)element, it must be unique.

Exercise 2. *Let $(S, \leq)$ be a partial order let A be a non-empty subset.*

1. *If a, a' are two minimum (or maximum) elements in A then $a = a'$. (Hence, we say a is* **the** *minimum (or maximum) element in A.*

2. *If a is the minimum element in A, then a is the unique minimal element in A.*

3. *If A has a maximum element a, then a is also the minimum element of $UB(A)$.*

Let $(S, \leq)$ be a partial order. Let $\emptyset \neq A \subseteq S$. $s \in S$ is the **least upper bound (or supremum)** of A (written $s = \sup(A)$) if s is the minimum element in $\text{UB}(A)$. $\sup(A)$ is undefined if $\text{UB}(A)$ is empty or has no minimum element. Similarly, when $\text{LB}(A)$ is non-empty and has a maximum element, then the element is called the *greatest lower bound (or infimum)* of A, (written $\inf(A)$).

Before discussing the next example, we note the following property of real numbers.

Fact 1. *Let $a, b \in \mathbf{R}$ such that $a < b$. Then there exists a rational number q such that $a < q < b$.*

Example 2. *The set $A = \{x : x^2 \leq 2\}$ in $(\mathbf{R}, \leq)$ has a unique maximal element $\sqrt{2}$ (which therefore is the maximum element) and unique minimal element $-\sqrt{2}$ (which is the minimum element). Further $\inf(A) = -\sqrt{2}$ and $\sup(A) = \sqrt{2}$.*

Exercise 3. *Show that in $(\mathbf{Q}, \leq)$, the set $A = \{x : x^2 \leq 2\}$ has no supremum, infimum, maximum or minimum. (Hint: Use the fact stated earlier.)*

Exercise 4. *In the partial order $(S, \leq)$, let $\emptyset \neq A \subseteq S$ be given. In each of cases below, find UB(A), LB(A), $\sup(A)$, $\inf(A)$, and all minimal and maximal elements of A, whenever they exist.*

1. *$S = \mathbf{R} \times \mathbf{R}$. Let $\leq$ be defined as earlier. Let $A = \{(x, y) : x^2 + y^2 = 1\}$.*
2. *In the same partial order as above, $A = \{(x, y : |x| + |y| = 1\}$*
3. *In $(\mathbf{N}, |)$, $|$ is the divisibility relation, let a, b be positive integers. $A = \{x \in \mathbf{N} : x = ai + bj$ for some $i, j \in \mathbf{Z}\}$.*
4. *In $(\{0, 1\}^n, \leq)$, $A = \{e_i :$ where e_i is the vector with 1 in the i^{th} position and 0 everywhere else, $1 \leq i \leq n\}$*

A partial order $(S, \leq)$ is a **lattice** if for every *finite* non empty subset A of S, $\sup(A)$ and $\inf(A)$ exists. A lattice is a **complete lattice** if for every non-empty subset A of S (finite or infinite) $\sup(A)$ and $\inf(A)$ exists.

Let $(S, \leq)$ be a lattice. Let $\emptyset \neq A \subseteq S$. A is a **bounded** lattice if both $\text{UB}(A)$ and $\text{LB}(A)$ are non-empty. The lattice $(S, \leq)$ is bounded if S is bounded. It is not hard to see that a lattice is bounded if and only if $\sup(S)$ and $\inf(S)$ exists (prove!). In a bounded lattice, the symbols $\top$ and $\perp$ are sometimes used to denote $\sup(S)$ and $\inf(S)$. In a bounded lattice $\top$ must be the maximum element and $\perp$ must be the minimum element (why?). It is also clear that any complete lattice must be bounded (why?).

Exercise 5. *Show that the lattices $(\mathbf{R}, \leq)$, $(\mathbf{Q}, \leq)$ and $(\mathbf{N}, \leq)$ and $(\mathbf{N}, |)$ are not complete lattices*

Sometimes the reason for the "incompleteness" of a lattice is that $\top$ and $\perp$ does not exist. In such cases by artificially adding $\top, \perp$, we may be able to make the lattice complete.

Definition 1. *The extended real number, rational number and natural number system are defined by* $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$, $\overline{\mathbf{Q}} = \mathbf{Q} \cup \{\pm\infty\}$ *and* $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$, *where* $\pm\infty$ *satisfies* $-\infty \leq x$ *for all* x *and* $+\infty$ *satisfies* $x \leq +\infty$ *for all* x *in the respective sets. (we may write* ∞ *for* $+\infty$ *hereafter.)*

Fact 2. *Let* A *be any bounded subset of* $(\mathbf{R}, \leq)$. *Then* $\sup(A)$ *and* $\inf(A)$ *exists.*

Exercise 6. *Which among the lattices* $(\overline{\mathbf{R}}, \leq)$, $(\overline{\mathbf{Q}}, \leq)$, $(\overline{\mathbf{N}}, \leq)$, $(\overline{\mathbf{N}}, |)$ *are complete? (Assume the Fact above to reason about* $\overline{\mathbf{R}}$*).*

Let $(S, \leq)$ be a lattice. Let $a, b \in S$. The notation $a \vee b$ and $a \wedge b$ will be used to denote $\sup\{a, b\}$ and $\inf\{a, b\}$ respectively. The following exercises develop some properties of $\vee$ and $\wedge$.

Observation 1. *The following properties are immediate from the definition:*

1. $a \leq b$ *if only if* $a \wedge b = a$ *if and only if* $a \vee b = b$.
2. *i)* $a \wedge b \leq a$. *ii)* $a \leq a \vee b$

Exercise 7. *Let* $(S, \leq)$ *be a lattice (not necessarily complete). Let* $a, b, c \in S$. *Prove the following. (The observations above will prove handy).*

1. *i)* $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *ii)* $(a \vee b) \vee c = a \vee (b \vee c)$

   Proof. Here is a proof for (i). Let $x = a \wedge b$ and $y = b \wedge c$. Required to prove that $x \wedge c = a \wedge y$. Let $t = x \wedge c$. Then $t \leq c$ and $t \leq x$ (by definition of $\wedge$). By the previous observation, from $t \leq x$ we can conclude that $t \leq a$ and $t \leq b$. From $t \leq b$ and $t \leq c$ we conclude $t \leq y$ (by the definition of $\wedge$.) Now, from $t \leq a$ and $t \leq y$ we conclude $t \leq a \wedge y$. This proves $x \wedge c \leq a \wedge y$. The other direction is proved Similarly. □

2. *i).* $a \wedge b = b \wedge a$ *ii).* $a \vee b = b \vee a$.
3. *i)* $a \wedge (a \vee b) = a$ *ii).* $a \vee (a \wedge b) = a$.
4. *Give examples for lattices where i)* $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$ *and ii)* $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$. *(Hint: Look for small lattices where this fails to hold. Lattices were the above equalities fail to hold are called* non-distributive *lattices.*