Department of Computer Science and Engineering, NIT Calicut

Lecture 4: Countable and Uncountable Sets

K. Murali Krishnan

If S is a finite set, the **cardinality** of S denoted by $|S|$ denotes the number of elements in S. In this lecture we extend the notion of cardinality to infinite sets. $\mathbf{N}, \mathbf{Q}$ and $\mathbf{R}$ will be used to denote the set of natural numbers, rational numbers and real numbers.

Definition 1. *Let S and T be arbitrary sets. S and T are said to have the same cardinality (written $|S| = |T|$) if there exists a bijection from S to T. We say $|S| \leq |T|$ if there exists an injective map from S to T.*

The following are some trivial consequences of the definition.

Exercise 1. *Let S, T, U be arbitrary sets.*

1. *If $|S| = |T|$, then both $|S| \leq |T|$ and $|T| \leq |S|$ holds.*
2. *If $|S| = |T|$ and $|T| = |U|$ then $|S| = |U|$.*
3. *If $|S| \leq |T|$ and $|T| \leq |U|$ then $|S| \leq |U|$.*

Given two arbitrary sets S and T, it is unclear whether if $|S| \leq |T|$ and $|T| \leq |S|$ holds true then $|S| = |T|$ is true or not. This, in fact, is non-trivial to prove. It is also non-trivial to prove that one among the relations $|S| \leq |T|$ or $|T| \leq |S|$ must be true.

Intuitively, two sets of the same cardinality have the "same number of elements". The following examples demonstrate some counter-intuitive properties of infinite sets.

Exercise 2. *Prove the following:*

1. *The set $2\mathbf{N} = \{0, 2, 4, \ldots\}$ satisfies $|2\mathbf{N}| = |\mathbf{N}|$.*
2. *There is an injective map from any set S to its power set 2^S. Hence $|S| \leq |2^S|$.*
3. *The map $f : \mathbf{N} \times \mathbf{N} \mapsto \mathbf{N}$ defined by $f(x, y) = 2^x(2y + 1) - 1$ is a bijection. Hence, $|\mathbf{N} \times \mathbf{N}| = |\mathbf{N}|$. Try to find a different bijection.*
4. *Let $[a, b]$ and $[c, d]$ be distinct intervals on the real number for some $a < b$ and $c < d$. Show that $|[a, b]| = |[c, d]|$. Show that the map $f(x) = c + (\frac{d-c}{b-a})x$ is a bijection between the two intervals. Try to find a different bijection.*
5. *$|\mathbf{Q}| \leq |\mathbf{N} \times \mathbf{N}|$. Use this to conclude that both $|\mathbf{Q}| \leq |\mathbf{N}|$ and $|\mathbf{N}| \leq |\mathbf{Q}|$ holds.*

Definition 2. *A set S is countably infinite if $|\mathbf{N}| = |S|$; that is, if there exists a bijection from S to the set of natural numbers. A set is countable if it is either finite or countably infinite. A set is uncountably infinite or uncountable if it is not countable.*

Exercise 3. *Show that if S, T are two disjoint countable sets then $S \cup T$ is countably infinite.*

Theorem 1. *Let $S_0, S_1, \ldots$ are disjoint countably infinite sets, then their union is countably infinite.*

Proof. Without loss of generality, we may write $S_0 = \{s_0(0), s_0(1), s_0(2), \ldots\}$, $S_1 = \{s_1(0), s_1(1), s_1(2) \ldots\}$,$S_2 = \{s_2(0), s_2(1), s_2(2) \ldots\}$ etc. (why?) Define the following function $f : \mathbf{N} \mapsto \bigcup_{i \in \mathbf{N}} S_i$ as follows:
$f(0) = s_0(0),\ f(1) = s_0(1), f(2) = s_1(0),\ f(3) = s_0(2), f(4) = s_1(1), f(5) = s_2(0),\ f(6) = s_0(3), f(7) = s_1(2), f(8) = s_2(1), f(9) = s_3(0) \ldots$. It is clear that for each $s_i(j)$ there exists a unique $n \in \mathbf{N}$ such that $f(n) = s_i(j)$. Conversely, for each $n \in \mathbf{N}$, there exists unique i, j such that $f(n) = s_i(j)$. □

Exercise 4. *In the above proof, find an explicit closed form formula for the function $f^{-1}(s_i(j))$*

Exercise 5. *Show that the collection of all finite subsets of* $\mathbf{N}$ *is countably infinite.*

The next theorem shows that uncountable sets exist. The proof technique used is called *Diagonal argument*

Theorem 2. *Let S be any non-empty set. There is no bijection from S to its power set, 2^S.*

Proof. Assume that there exists a bijection from S to 2^S. Then, for each $x \in S$, $f(x)$ is a subset of S. Consider the subset T of S defined as: $T = \{x : x \notin f(x)\}$. As f is surjective, there must be some $x \in S$ such that $T = f(x)$. But this leads to the following contradiction: $x \in f(x)$ if and only if $x \in T$ if and only if $x \notin f(x)$ (the first equivalence follows by the assumption that $f(x) = T$, the second by the definition of T). □

A consequence of the above theorem is that $2^{\mathbf{N}}$ is not countable.

Exercise 6. *Show that the collection of all infinite subsets of* $\mathbf{N}$ *is uncountable. (Hint: Use the previous exercise)*

Exercise 7. *Let $S_0, S_1, S_2 \ldots$ be a sequence of subsets of $\mathbf{N}$. Define a set T such that $T \neq S_i$ for any $i \in \mathbf{N}$. Use this observation to give a direct proof that $2^{\mathbf{N}}$ is uncountable.*

Exercise 8. *Consider any collection $F = \{f_0, f_1, f_2, \ldots\}$ of functions from $\mathbf{N}$ to $\mathbf{N}$. Construct a function $g : \mathbf{N} \mapsto \mathbf{N}$ such that $g \neq f_i$ for every $i \in \mathbf{N}$. The existence of such g shows that any countable collection of functions from $\mathbf{N}$ to $\mathbf{N}$ is incomplete (in the sense we can find a function from $\mathbf{N}$ to $\mathbf{N}$ that is not present in the list). Hence argue that the set of all function from $\mathbf{N}$ to $\mathbf{N}$ is uncountable.*

It is easy to see that the set of all functions from $\mathbf{N}$ to $\{0, 1\}$ is uncountable. This gives another proof that $2^{\mathbf{N}}$ is uncountable (why?).

Our next objective is to show that the number of points in the real line $(0, 1]$ is uncountably infinite. Consider the interval on the real line $(0, 1]$. Each real number x greater than 0 and less than 1 has a unique infinite **binary** expansion of the form $x = 0.x_0x_1x_2x_3 \ldots$. (Rational numbers will have two expansions - one terminating and one non-terminating. Here we take the infinite one. For example, if $x = \frac{1}{2}$, $x = 0.1 = 0.011111\ldots$ and the latter expansion is taken).

Exercise 9. *Let* $x^0 = 0.x^0_1x^0_2x^0_3\ldots$, $x^1 = 0.x^1_0x^1_1x^1_2\ldots$, $x^3 = 0.x^3_0x^3_1x^3_2\ldots$ *be a any collection of infinite binary expansion sequences. Construct a new expansion sequence* $y = 0.y_0y_1y_2\ldots$ *such that* $y_i \neq x^i_i$ *for any* $i \in \mathbf{N}$*. Use this to conclude that the set of all non-terminating binary expansions is uncountably infinite.*

Exercise 10. *Let* S *be any infinite subset of natural numbers. We can associate an* **non-terminating** *binary expansion* x^S *associated with* S *as follows:* $x^S = x^S_0, x^S_1, x^S_2, \ldots$ *where* $x^S_i = 1$ *if* $i \in S$ *and* $x^S_i = 0$ *if* $i \notin S$*. Show that this association is a one one mapping from the set of all infinite subsets of* $\mathbf{N}$ *to the set of all non-terminating binary expansions. Argue that the set of all non-terminating binary expansions have the same cardinality as the collection of all infinite subsets of* $\mathbf{N}$*.*

Exercise 11. *Show that the set of all finite subsets of subsets of* $\mathbf{N}$ *has the same cardinality as the set of all finite binary expansions. Hence argue that the set of all binary expansions have the same cardinality as* $2^{\mathbf{N}}$*.*

Exercise 12. *Let* S, T *be disjoint sets with* S *countable and* T *uncountable. Show that* $S \cup T$ *is uncountable.*

The following property assumed about sets, is called the **axiom of choice**.

Fact 1 (Axiom of Choice)**.** *Let* I *be any non-empty set. Let* $\{A_i\}_I$ *be an arbitrary collection of non-empty sets. Then* $\prod_{i \in I} A_i \neq \emptyset$*. Another way of stating the axiom is to say that we can assume that there exists a function that for each* $i \in I$ *gives a value in* A_i*, i.e.,* $\exists a : I \mapsto \bigcup_{i \in I} A_i$ *such that* $a(i) \in A_i$*.*

Theorem 3. *Let* S, T *be arbitrary non-empty sets. There exists an injective function from* S *to* T *if and only if there exists a surjective function from* T *to* S*.*

Proof. Let $f : S \mapsto T$ be injective. Let $s \in S$ be chosen arbitrarily. Define $g : T \mapsto S$ as $g(t) = f^{-1}(t)$ if $t \in f(S)$ and $g(t) = s$ if $t \in T - f(S)$ is a surjective map from T to S. The converse uses axiom of choice. Let g be a surjective map from T to S. The collection of inverse images g for each point in S, $\{g^{-1}(s)\}_{s \in S}$ is a partition of T. By axiom of choice, there exists a function $f : S \mapsto T$ with $f(s) \in g^{-1}(s)$ for each $s \in S$. This function is injective (why?). $\square$