Department of Computer Science and Engineering, NIT Calicut

Lecture 4: Axioms and Models

Prepared by: K. Murali Krishnan

In a FOLG(=) formula of the form $\forall x\phi$ or $\exists x\phi$, **change of variable** of $x \in X$ with another variable y without affecting the semantics of the formula is possible, provided: (i) y does not occur in ϕ and (ii) we replace all occurrence of x in the scope of the quantifier $\forall x$ (or $\exists x$) uniformly with y. Suppose we define a *primitive* formula in $FOLG(=)$ as follows: $R(x,y), (x = y), \neg R(x,y), \neg(x = y)$ are primitive for all $x, y \in X$. If ϕ is primitive, so are $\phi \vee \psi$ and $\phi \wedge \phi$. Essentially a primitive formula is a **boolean expression** created out of the basic terms $R(x,y), (x = y), \neg R(x,y), \neg(x = y)$ using just $\vee$ and $\wedge$ as combining operators. Note that for any formula ϕ in $FOLG(=)$, we can convert ϕ to a form where all the quantifiers $(\forall, \exists)$ appear first, and the rest of the formula is a primitive formula. Such a re-written form is called a **prenex normal form** re-writing of the formula. The primitive part of a formula is prenex form called the **matrix** of the prenex formula.

Example 1. $\forall x(\forall y R(x,y) \rightarrow \exists y R(y,x))$. *If we substitute y with z in the scope of* $\exists y R(y,x)$ *we get the equivalent formula* $\forall x(\forall y R(x,y) \rightarrow \exists z R(z,x))$. *The formula is equivalent to* $\forall x(\neg\forall y R(x,y) \vee \exists z R(z,x))$*, which in turn is equivalent to* $\forall x[\exists y \neg R(x,y) \vee \exists z R(z,x)]$*. Using the rules in the following lemma, we can simplify this expression further to* $\forall x \exists y \exists z[\neg R(x,y) \vee R(z,x)]$*, which is in the prenex form with matrix* $\neg R(x,y) \vee R(z,x)$*.*

First order logic yields additional tautological implications and equivalences involving quantifiers, some of which are listed below.

Lemma 1. *Let* ϕ, ψ *are closed formulas in* $\mathcal{F}$*,* $x, y \in X$*.*

1. $\forall x(\phi \wedge \psi) \Leftrightarrow \forall x\phi \wedge \forall x\psi$.
2. $\forall x\phi \vee \forall x\psi \Rightarrow \forall x(\phi \vee \psi)$.
3. $\exists x\phi \vee \exists x\psi \Leftrightarrow \exists x(\phi \vee \psi)$
4. $\exists x(\phi \wedge \psi) \Rightarrow \exists x\phi \wedge \exists x\psi$
5. $\neg\forall x\phi \Leftrightarrow \exists\neg\phi$
6. $\neg\exists x\phi \Leftrightarrow \forall\neg\phi$

Proof. The proofs follow directly from the semantic definitions. To prove (1), Suppose $G \models \forall x(\phi \wedge \psi)$. Then, for every $\tau : X \rightarrow V$, $(G, \tau) \models \phi \wedge \psi$. Thus for all τ, $(G, \tau) \models \phi$ and $(G, \tau) \models \psi$. Hence $G \models \forall x\phi$ and $G \models \forall x\psi$. Conversely, Suppose $G \models \forall x\phi$ and $G \models \forall x\psi$. Then, for each τ, $(G, \tau) \models \phi$ and $(G, \tau) \models \psi$. Hence, every τ must satisfy $(G, \tau) \models \phi \wedge \psi$. Other results are proved similarly. □

Definition 1. *Let G be a graph. Define* $\mathcal{F}(G) = \{\phi \in \mathcal{F} : G \models \phi\}$ *as the collection of all formulas which are true in G. If* $\mathcal{G}$ *is a collection of graphs, define* $\mathcal{F}(\mathcal{G}) = \bigcap_{G \in \mathcal{G}} \mathcal{F}(G)$ *as the collection of all formulas which hold for all graphs in* $\mathcal{G}$*.*

Example 2. *Consider the graph* $G = (V, E)$ *with* $V = \{0, 1, 2, 3, ...\}$ *and* $E = \{(i, i+1) : i \geq 0\}$*. Each vertex in the graph has out degree exactly one. Hence, the following properties are true in* G. $\forall x \exists y G(x, y)$ *(every vertex has an outgoing edge).* $\forall x \forall y \forall z (G(x, y) \wedge G(x, z) \rightarrow (y = z))$ *(a vertex has at most one out-going edge)* $\forall x \forall y \forall z (G(x, z) \wedge G(y, z) \rightarrow (x = y))$ *(a vertex has at most one in-coming edge). These formulas are in* $\mathcal{F}(G)$*. What can you say about finite graphs satisfying these properties?*

Exercise 1. *Give a categorical collection of axioms* $\mathcal{A}$ *such that* $\mathcal{M}(\mathcal{A}) = \{G\}$ *(up to isomorphism) for the following graph* $G = (V = \{1, 2\}, E = \{(1, 2), (1, 1), (2, 2)\})$*.*

Exercise 2. *Write down a collection of axioms to express the following properties: (i) There is exactly one vertex with out-degree zero and in-degree one. (ii) There is exactly one vertex with in-degree zero and out-degree one. (iii) Every other vertex has both in-degree and out-degree exactly one. What can you conclude about finite graphs satisfying these properties? Show that there are graphs with infinitely many vertices satisfying all these properties.*

Just as with propositional logic, we often postulate the defining properties of a first order system (in our context, a class of graphs satisfying certain properties - for instance, the class of graphs representing equivalence relations) as a set of axioms, say $\mathcal{A}$. This leads to two decision problems: 1. Given a graph G, does G satisfy $\mathcal{A}$. 2. Given a formula ϕ, is ϕ a logical consequence of $\mathcal{A}$. The former problem is easier, and is solvable in polynomial time in the number of vertices of the graph. The latter is much harder and is unsolvable in a general. As with propositional logic, classical deduction methods exists for first order logic as well.

Definition 2. *Let* $\mathcal{A} \subseteq \mathcal{F}$*, define* $Th(\mathcal{A}) = \{\phi : \mathcal{A} \models \phi\} = \{\phi : G \models \phi$ *for every* $G \in \mathcal{M}(\mathcal{A})\}$*.* $Th(\mathcal{A})$ *is the set set of all logical consequences (or "theorems") that follow from the "axiom set"* $\mathcal{A}$*.*

Definition 3. *Let* $\mathcal{G}$ *be any collection of graphs. A collection of FOLG(=) formulas* $\mathcal{A}$ *is said to axiomatically characterize* $\mathcal{G}$ *if* $\mathcal{M}(\mathcal{A}) = \mathcal{G}$*.*

Exercise 3. *Find two non-isomorphic graphs satisfying the following axioms. Argue that no graph with finite number of vertices can satisfy the axioms. A1.) Every vertex has out-degree one. A2.) There is exactly one vertex with in-degree zero, and every vertex except this vertex has in degree 1.*

Exercise 4. *In the previous exercise, suppose the second property is changed to: every vertex has out degree one, what can you say about finite models satisfying the axioms? Give two non-isomorphic models with six vertices.*

Exercise 5. *Consider the following properties: A1) There exists exactly one vertex with in degree zero, every other vertex has in degree exactly one. A2) Every vertex has out degree exactly two. Give two non isomorphic graphs satisfying the above properties. (This is non-trivial).*

Exercise 6. *Write down the axioms to express the condition that* G *represents the graph of a total (linear) order. (That is, the relation is reflexive, symmetric, transitive and any two elements are comparable).*

Exercise 7. *To the axioms of linear order, add the property (called density propery):* $\forall x \forall y \exists z [R(x,z) \wedge R(z,y)]$. *Give four non-isomorphic countable models satisfying all the axioms. Argue that no finite model satisfy all the axioms.*

1 Peano's Axioms

In an effort to axiomize natural numbers axiomatically, we may identify natural numbers with the graph $(\mathcal{N}, E)$ where $E = \{(0,1),(1,2),(2,3),\ldots\}$ and try to write down a collection of axioms $\mathcal{A}$ such that whenever $G \in \mathcal{M}(\mathcal{A})$, $G \cong \mathcal{N}$. Here we consider a variant of FOLG(=) by adding the special symbol 0. We extend the syntax of FOLG(=) expressions and permit expressions like $R(0,x)$, $x = 0$ etc. When considering models (graphs) we assume that the graph contain a pre-named vertex with name 0, that is $0 \in V(G)$ for any graph G. With this terminology, the expression $\forall x \neg R(x,0)$ expresses the condition that the vertex 0 has in-degree 0. The addition of 0 is not necessary in that its properties can be axiomized (see previous exercises), but the use of 0 makes expressions less cumbersome.

In 1889, Italian mathematican Giuseppe Peano proposed for axiomizing $\mathcal{N}$. In the description below, we call vertex v a successor of a vertex u in a graph G if $(u,v) \in E(G)$.

1. Every vertex has at least one successor.
2. Every vertex has only one successor.
3. No two vertices can have the same vertex as successor.
4. No vertex has 0 as the successor.
5. Let S be a set of vertices satisfying the following conditions: a) $0 \in S$, b) if $u \in S$ and $(u,v) \in E$ then $v \in S$. Then, S contains every vertex in G.

It is easy to see that the first four properties can be easily described in FOLG(=). It turns out that the last property (called the **Principle of Mathematical Induction**) cannot be formulated in FOLG(=) and would need the more powerful second order logic for proper formulation.