Your Name and Roll No:

Instruction: Only answers written in the space provided will be evaluated.

1. On the set $\mathbf{Z}_{30}$, define the equivalence relation R by aRb if and only if GCD$(a, 30)$ =GCD$(b, 30)$. What is the *index* (number of equivalent classes) of R? Write down all the equivalence classes under R. **Ans:** [3]

   Soln: The partitioning induced by the relation is $\{\{1, 7, 11, 13, 17, 19, 23, 29\}, \{2, 4, 8, 14, 16, 22, 26, 28\}, \{3, 9, 21, 27\}, \{5, 25\}, \{6, 12, 18, 24\}, \{10, 20\}, \{15\}, \{0\}\}$. As there are eight classes, the index is 8.

2. Give an example for a monotone function from $\mathbf{Q}$ to $\mathbf{Q}$ that has no fix point in $\mathbf{Q}$. You must argue correctness. (Hint: The example done in class for the interval $[1, 2] \cap \mathbf{Q}$ can be extended to a monotone function on the whole of $\mathbf{Q}$ with a bit of thought!) [3]

   Soln: The simplest way to do this is to extend the function which we saw in the class to the whole of $\mathbf{Q}$ without losing monotonicity and continuity. Here is one possible way: define $f(x) = x + \frac{1}{4}(2 - x^2)$ for $x \leq 2$ and $f(x) = \frac{3}{2}$ for $x \geq 2$.

3. Let $\mathbf{S}$ be the collection all boolean functions on integers. That is $S = \{f : \mathbf{N} \mapsto \{0, 1\}\}$ Let $f_0, f_1, f_2, \ldots$ be a collection of functions from S. [3+1]

   1. Define a function $g : \mathbf{N} \mapsto \{0, 1\}$ such that $g \neq f_n$ for any $n \in \mathbf{N}$. (with a proof that $g \neq f_n$ for all $n \in \mathbf{N}$).

      Soln: Define $g(n) = 1 - f_n(n)$. If $g = f_k$ for some k, then we have $g(k) = f_k(k)$ (as $g = f_k$). However, by the definition of g we also have $g(k) = 1 - f_k(k)$. Combining the two conditions we get $f_k(k) = 1 - f_k(k)$, which is a contradiction since f_k can take only values 0 or 1.

   2. What conclusion can you draw from the existence of such a function? **Ans:**

      Soln: The argument shows that there is no bijection from $\mathbf{N}$ to S. In other words, S is not countably infinite.

4. Let $(S, \leq)$ be a complete lattice. Assume $\top$ and $\bot$ and the maximum and minimum elements in S. Let $f : S \mapsto S$ be a monotone function on S. Let $A = \{x : f(x) \leq x\}$. [1+4]

   1. Is it always true that $LB(A) \neq \emptyset$? Justify **Ans:**

      Soln: First A is non-empty because $f(\top) \leq \top$. Clearly $\bot \in LB(A)$ by definition of $\bot$. Hence $LB(A)$ cannot be empty.

   2. Is it true that $\inf(A)$ a fix point of f? If so give a proof. Otherwise give an example for a lattice where this statement fails. (Answer on the reverse sheet)

      Soln: Let $a_0 = \inf(A)$. Such A must exist because A is non-empty (see argument above) and S is a complete lattice. For each $a \in A$, $a_0 \leq a$. By monotonicity of f, we conclude that $f(a_0) \leq f(a) \leq a$. for each $a \in A$ (the latter inequality is a defining property of any $a \in A$). From this, we conclude that $f(a_0) \in LB(A)$. Since a_0 is the greatest lower bound of A, we have $f(a_0) \leq a_0$. Now, to prove that $a_0 \leq f(a_0)$, it suffices to prove that $f(a_0) \in A$. This is proved as follows. We have shown above that $f(a_0) \leq a_0$. By monotonicity of f, $f(f(a_0)) \leq f(a_0)$. This inequality suffices to prove that $f(a_0) \in A$ by the definition of A.