CS 2006: Discrete Structures **Mid II Part II** **March 2018**

Answer strictly in the space provided. **Name:**

1. Give two clauses C_1, C_2 such that neither C_1 nor C_2 is a tautology, but their resolvent is a tautology. [2]

   Soln: $C_1 = (p \vee q)$, $C_2 = (\neg p \vee \neg q)$. There are two resolvents: $(p \vee \neg p)$ and $(q \vee \neg q)$, both being tautologies.

2. Let $\{C_1, C_2, \ldots, C_n\}$ be a collection of propositional clauses. Suppose the resolvent of C_1 and C_2 is a tautology, can we conclude that $\{C_1, C_2, \ldots, C_n\}$ is satisfiable? Prove / Give counter example. [2]

   Soln: Consider $C_1 = (p \vee q)$, $C_2 = (\neg p \vee \neg q)$, $C_3 = (p \vee \neg q)$, $C_2 = (\neg p \vee q)$. Clearly $\{C_1, C_2, C_3, C_4\}$ is unsatisfiable. But both resolvents of C_1 and C_2 are tautologies as seen in the previous question.

3. A triangle-free graph is one that does not contain (directed) cycles of length 3. Write down FOLG(=) axioms to characterize all triangle-free graphs (finite and infinite). [2]

   Soln: $\forall x \forall y \forall z[(x \neq y) \wedge (y \neq z) \wedge (x \neq z) \wedge G(x, y) \wedge G(y, z) \rightarrow \neg G(z, x)]$.

4. Is it possible to give an FOLG(=) characterization for all (finite and infinite) graphs with the following property: whenever there is a path from one vertex to another, then there is also an edge from the first to the other. (Either write down axioms or prove no such characterization exists). [2]

   Soln: The property stated is nothing but transitivity. $\forall x \forall y \forall z[G(x, y) \wedge G(y, z) \rightarrow G(x, z)]$.

5. Is it possible to write down a set of FOLG(=) axioms that characterize all (finite and infinite) graphs without cycles. (Either write down axioms or prove no such characterization exists). [2]

   Soln: The following (infinite) set of axioms suffices: 1. $\phi_1 = \forall x \neg G(x, x)$.
   2. $\phi_2 = \forall x \forall y[G(x, y) \rightarrow \neg G(y, x)]$. 3. $\phi_3 = \forall x \forall y \forall z[G(x, y) \wedge G(y, z) \rightarrow \neg G(z, x)], \ldots$

6. A finite bipartite graph is one that contains only finitely many vertices, and does not contain any odd length cycles. Is it possible to have an FOLG(=) characterization for bipartite graphs? (Either write down axioms or prove no such characterization exists). [2]

   Soln: You may use the Skolem Lowenheim theorem to argue that FOLG(=) characterization for finite bipartite graphs is not possible. Here is another direct argument Consider the infinite collection of formulas: 1. $\phi_2 = \exists x \exists y \neg(x = y)$. 2. $\phi_3 = \exists x \exists y \exists z[\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z)] \ldots$
   Let $\mathcal{B} = \{\phi_2, \phi_3, \ldots\}$. $\mathcal{B}$ has no finite models (why?).

   Now, Suppose that $\mathcal{A}$ be a collection of formulas that characterize all finite bipartite graphs. Any model satisfying $\mathcal{A} \cup \mathcal{B}$ will be infinite (and will be a model for $\mathcal{A}$ as well). Hence, if we prove that $\mathcal{A} \cup \mathcal{B}$ is satisfiable, then it will follow that $\mathcal{A}$ has an infinite model, and hence cannot be a characterization for finite bipartite graphs.

   But satisfiability of $\mathcal{A} \cup \mathcal{B}$ is easy to prove using the compactness theorem. Any finite subset of B has finite models (why?). Hence every finite subset of $\mathcal{A} \cup \mathcal{B}$ too must have finite models (why?). Hence, by compactness theorem, $\mathcal{A} \cup \mathcal{B}$ must be satisfiable.

7. Let $\mathcal{A} = \{p_1 \vee \neg p_2, \neg p_1 \vee p_2, p_2 \vee \neg p_3, \neg p_2 \vee p_3, p_3 \vee \neg p_4, \neg p_3 \vee p_4, \ldots\}$. a) Is $\mathcal{A}$ consistent? b) Is $\mathcal{A}$ categorical? c) Does there exist a formula independent of $\mathcal{A}$ over $\{p_1, p_2, p_3, \ldots\}$ (either prove no such formula exists or give an independent formula). Answer on the reverse side. [3]

   Soln: This question is easy. $p_1 = T, p_2 = T, p_3 = T, \ldots$ and $p_1 = F, p_2 = F, p_3 = F, \ldots$ are two satisfying truth assignments to $\mathcal{A}$. Hence $\mathcal{A}$ is both **consistent**, and **not categorical**. Since the first truth assignment satisfies p_1 whereas the second one satisfies $\neg p_1$, the atomic formula p_1 is **independent** of $\mathcal{A}$.