Problem Set I

1. Consider the vectors $B = \{b_1 = (1,1,1), b_2 = (1,1,0), b_3 = (2,2,1)\}$ in $\mathbf{R}^3$. □

   1. Show that these vectors are not linearly independent.
   2. Find two linearly independent vectors v_1 and v_2 different from the vectors in B such that both v_1 and v_2 are in $span(B)$.
   3. Find two linearly independent vectors w_1 and w_2 such that both w_1 and w_2 are not in $span(B)$.
   4. Find a vector v such that $\{b_1, v\}$, $\{b_2, v\}$ and $\{b_3, v\}$ are linearly independent sets, but $\{b_1, b_2, v\}$ is linearly dependent.
   5. Is the vector $w = (1,1,2)$ in $span(B)$? If so, find scalars (real numbers) α, β, γ such that $w = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3$. Otherwise, show that no such scalars $\alpha_1, \alpha_2, \alpha_3$ exists.
   6. Show that the vector $u = (3,3,2)$ can be written as $u = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 = \alpha_1' b_1 + \alpha_2' b_2 + \alpha_3' b_3$ such that $(\alpha_1, \alpha_2, \alpha_3) \neq (\alpha_1', \alpha_2', \alpha_3')$.

2. Consider the vectors $B = \{b_1 = (1,0,1), b_2 = (1,1,0), b_3 = (0,1,1)\}$ in $\mathbf{R}^3$. □

   1. Show that the vectors b_1, b_2, b_3 are linearly independent.
   2. Find scalars $\alpha_1, \alpha_2, \alpha_3$ such that $v = (1,1,1) = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3$.
   3. Show that for any vector $w = (x, y, z) \in \mathbf{R}^3$, we can find scalars $\alpha_1, \alpha_2, \alpha_3$ such that $w = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3$. Express α_1, α_2 and α_3 in terms of x, y and z.
   4. Suppose $v = (1,1,1) = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 = \alpha_1' b_1 + \alpha_2' b_2 + \alpha_3' b_3$ then show that $(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1', \alpha_2', \alpha_3')$.
   5. Does B form a basis for $\mathbf{R}^3$?

3. Suppose a vector $v \in \mathbf{R}^3$ has coordinates $(1,1,1)$ w.r.t the basis $(1,1,0), (1,0,1), (0,1,1)$, Find its coordinates with respect to the basis $(1,0,0), (1,1,0), (1,1,1)$. □

4. Consider the set of all polynomials with real coefficients with degree at most 3. (This set is called $\mathbf{R}_3[x] = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, a_1, a_2, a_3 \in \mathbf{R}\}$). □

   1. Show that $\mathbf{R}_3[x]$ is a vector space over $\mathbf{R}$.
   2. Show that the polynomials $e_1(x) = 1, e_2(x) = x, e_3(x) = x^2, e_4(x) = x^3$ is a basis for $\mathbf{R}_3[x]$. $(e_1(x), e_2(x), e_3(x), e_4(x))$ is called the standard basis of $\mathbf{R}_3[x]$ (why!?).
   3. Show that the polynomials $b_1(x) = 1, b_2(x) = \frac{x}{1!}, b_3(x) = \frac{x^2}{2!}, b_4(x) = \frac{x^3}{3!}$ form a linearly independent set (and hence a basis) in $\mathbf{R}_3[x]$.
   4. Find the coordinates of the polynomial $1 + x + x^2 + x^3$ with respect to the basis $(b_1(x), b_2(x), b_3(x), b_4(x))$
   5. Show that the polynomials $c_1(x) = 1, c_2(x) = \frac{x-1}{1!}, c_3(x) = \frac{(x-1)^2}{2!}, c_4(x) = \frac{(x-1)^3}{3!}$ forms a linearly independent set (and hence a basis) in $\mathbf{R}_3[x]$.
   6. Suppose a polynomial $p(x)$ has coordinates $(1,1,1,1)$ with respect to the basis $(b_1(x), b_2(x), b_3(x), b_4(x))$, what will be its coordinates with respect to a) basis $(e_1(x), e_2(x), e_3(x), e_4(x))$? b) basis $(c_1(x), c_2(x), c_3(x), c_4(x))$?

5. Consider the vector space of all polynomials of degree at most n over $\mathbf{R}$. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$. Find the co-ordinates of $f(x)$ over the basis $[1, (x-t), (x-t)^2, \cdots, (x-t)^{n-1}]$. (Hint: Evaluate $\frac{d^k}{dx^k} f(x)$ at $x = t$.) □

6. Consider the vector space of all polynomials over $\mathbf{R}$ (called the set $\mathbf{R}[x]$). Let t be any real number. Show that the set $B = \{1, \frac{(x-t)}{1!}, \frac{(x-t)^2}{2!}, \frac{(x-t)^3}{3!}, ...\}$ is a collection of linearly independent vectors in $\mathbf{R}[x]$. Argue that $span(B) = \mathbf{R}[x]$ and hence B is a basis of $\mathbf{R}[x]$. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$ be a polynomial in $\mathbf{R}[x]$. What will be cordinates of $f(x)$ with respect to the basis B? The representation of $f(x)$ with respect to the basis B is called the Taylor expansion of $f(x)$ with respect to the point t. □