## Problem Set IV

1. Let $V$ and $W$ be vector spaces of dimension $m$ and $n$ respectively. Let $c_1, c_2, \ldots c_m$ and $b_1, b_2, \ldots b_n$ be basis of $V$ and $W$. Let $T$ be a linear transformation from $V$ to $W$. Let $A$ be the matrix of $T$ w.r.t. basis $c_1, c_2, \ldots c_m$ (of $V$) and $b_1, b_2, \ldots b_n$ (of $W$). Prove the following: (Note: In an *if and only if* statement, there are two directions to be established!) ☐

    1. $T$ is injective if and only if $T(c_1), T(c_2), \ldots T(c_m)$ is a basis of $Image(T)$. (Thus, an injective linear map is an isomorphism between $V$ and $Image(T)$).
    2. $T$ is bijective if and only if $T$ is both injective and $m = n$.
    3. $T$ is injective if and only if $CRank(A) = m$. In particular, if $n < m$, $T$ cannot be injective.
    4. $T$ is surjective if and only if $Columnspan(A) = \mathbf{R}^n$.
    5. $T$ is bijective if and only if $m = n$ and $A$ is a non-singular matrix.

2. Let $e_1, e_2, \ldots$ represent the standard basis vectors (in $\mathbf{R}^n$). Consider the map $T : \mathbf{R}^3 \mapsto \mathbf{R}^2$ defined by $T(e_1) = e_2$, $T(e_2) = e_1$ and $T(e_3) = e_1 + e_2$. Let $c_1 = [0, 0, 1]^T$, $c_2 = [0, 1, 1]^T$, $c_3 = [1, 1, 1]^T$ and $b_1 = [1, 1]^T$, $b_2 = [1, -1]^T$. Find the matrix of $T$ with respect to basis $c_1, c_2, c_3$ and $b_1, b_2$. (Hint: In such problems, it often less cumbersome to find the expressions for $T(c_1), T(c_2)$ and $T(c_3)$ in terms of $b_1$ and $b_2$ and directly compute the matrix required, instead of using matrix formula $A' = BAC^{-1}$. However, the matrix formula is better suited for programming) ☐

3. Let $V$ be a vector space of dimension $n$. Let $b_1, b_2, \ldots b_n$ be a basis of $V$. Let $T$ be an operator on $V$. Let $A$ be the matrix of $T$ with respect to the basis $b_1, b_2, \ldots b_n$. Prove the following: ☐

    1. $T$ is bijective if and only if $T$ is injective if and only if $T$ is surjective. (Thus, proving either injectivity or surjectivity proves bijectivity for operators).
    2. $T$ is bijective if and only if columns of $A$ are linearly independent (and hence the volume of the parallelepiped generated by the columns, $det(A) \neq 0$).
    3. $T$ is bijective if and only if $0$ is not an Eigen value of $A$.

4. Consider the operator $T : \mathbf{R}^2 \mapsto \mathbf{R}^2$ whose matrix with respect to the standard basis $e_1, e_2$ is given by $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$. (I will simply write "consider the matrix $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$" instead of the above politically correct sentence in future!) What is the matrix of $A$ with respect to the basis of $b_1 = [1, 1]^T$, $b_2 = [1, -1]^T$. ☐

5. Suppose $T : V \mapsto V$ be a linear operator on the $n$ dimensional vector space $V$. Suppose $b_1, b_2, \ldots b_n$ be a basis of $V$ such that $T(b_1) = \lambda_1 b_1, T(b_2) = \lambda_2 b_2, \ldots T(b_n) = \lambda_n b_n$, where $\lambda_i$ is a scalar in $\mathbf{R}$ for $1 \leq i \leq n$. What will be the matrix of $T$ with respect to the basis $b_1, b_2, \ldots, b_n$? [A basis of $V$ with respect to which the matrix of a linear operator $T$ becomes a diagonal matrix is called a *diagonalizing basis* for $V$.] Find a diagonalizing basis for the matrix $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$. ☐

6. Consider the orthonormal basis $b_1 = [\frac{\sqrt{3}}{2}, \frac{1}{2}]^T$, $[\frac{1}{2}, -\frac{\sqrt{3}}{2}]^T$ of $\mathbf{R}^2$. Let $P_1$ be the orthogonal projection matrix to $b_1$ and let $P_2$ be the projection matrix to $b_2$. Consider the matrix $A = 2P_1 + 3P_2$. Find a diagonalizing basis for $A$. Let $T$ be the operator whose matrix is $A$ with respect to the standard basis of $\mathbf{R}^2$. Find the matrix of $T$ with respect to the diagonalizing basis you have found out for $A$. (Again, we will in future write "find the matrix of $A$ with respect to the diagonalizing basis" instead of "find the matrix of the operator $T$, whose matrix with respect to the standard basis is $A$, with respect to a basis that diagonalizes $T$"). ☐

7. Let $b_1, b_2, \ldots b_n$ be an orthonormal basis of $\mathbf{R}^n$. Consider the matrix $A = \lambda_1 b_1 b_1^T + \lambda_2 b_2 b_2^T + \cdots + \lambda_n b_n b_n^T$, where $\lambda_1, \lambda_2, \ldots \lambda_n$ are scalars in $\mathbf{R}$. Find a diagonalizing basis for $A$ (you may express the basis vectors as linear combination of $b_1, b_2, \ldots b_n$.) What is the matrix of $A$ with respect to this basis? ☐