Problem Set V

1. If A is a real symmetric $n \times n$ matrix such that $det(A) = 0$, then, show that the function $f : \mathbf{R}^n \times \mathbf{R}^n \mapsto \mathbf{R}$ defined by $f(u, v) = u^T Av$ does not define an inner product. Which inner product axiom is violated in this case? ☐

2. Let B be an $n \times n$ real orthogonal matrix (that is, a matrix that satisfies $B^T B = I$). Show that the columns of B are mutually perpendicular unit vectors with respect to the standard inner product of $\mathbf{R}^n$. ☐

3. In R^2 consider the positive matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. ☐

    1. Find the Eigen values A. Find an orthonormal basis for the Eigen space of each Eigen value of A.
    2. Find an orthogonal 2×2 matrix B of $\mathbf{R}^2$ such that $A = BDB^T$ where D is a diagonal matrix. (Writing a symmetric matrix in this way is called the spectral decomposition of A or the spectral factorization of A).

4. Let V be a real inner product space with inner product $()$. Let b be a unit vector. Define the projection (operator) along the direction b, P_b by $P_b(v) = (v, b)b$ for all $v \in V$. Find $Rank(P_b)$ and $Nullity(P_b)$. Suppose $b_1, b_2, \ldots b_n$ is an orthonormal basis of V and $b = \alpha_1 b_1 + \alpha_2 b_2 + \cdots \alpha_n b_n$. What will be the matrix of P_b with respect to the basis $b_1, b_2, \ldots b_n$? ☐

5. Let V be a real inner product space with inner product $()$. Suppose S is a subspace of dimension k. Let $b_1, b_2, \ldots, b_k$ be an orthonormal basis of S. Show that $P_S = P_{b_1} + P_{b_2} + \cdots + P_{b_k}$. That is, for all $v \in V$, $P_S(v) = (P_{b_1} + P_{b_2} + \cdots + P_{b_k})(v)$. Find $Rank(P_b)$ and $Nullity(P_b)$. This shows that projection into a subspace can be thought of as a (vector) sum of projections on to a collection of orthogonal directions. ☐

6. [Bessel's inequality] Let $b_1, b_2, \ldots b_k$ be orthogonal unit vectors in an n dimensional complex inner product space V with inner product function $()$. Let $v \in V$. Show that $||v||^2 \geq \sum_{i=1}^{k} P_{b_i}(v)^2$ where $P_i(v) = (v, b_i)b_i$ is the projection of v along the direction b_i. ☐

7. Let V be a real inner product space with inner product $()$. Suppose S is a subspace of dimension k. Let P_S be the projection function to the subspace S. Show that a) P_S satisfies $P_S^2 = P_S$ and b) P_S is symmetric - that is, for all $u, v \in V$, $(Pu, v) = (u, Pv)$. ☐

8. Let V be a real inner product space with inner product $()$. Suppose P is a linear operator on V. Suppose P satisfies a) $P^2 = P$ and b) P is symmetric. In this question, we will show that P is an orthogonal projection. Let $S = Image(P)$. ☐

    1. Show that if $s \in S$, $P(s) = s$. (Hint: You must use the fact that there exists some $v \in V$ such that $P(v) = s$).
    2. Show that $P(s) = s$ if and only if $s \in Image(P)$. (Thus, $S = Image(P)$ is precisely the Eigen space corresponding to Eigen value 1.)
    3. Prove that if $t \in S^\perp$ then $P(t) \in S^\perp$. Thus $S^\perp$ is P invariant. (This requires only use of the fact that P is symmetric. You do not need the property $P^2 = P$ for proving this).
    4. Show that if $t \in S^\perp$, $P(t) = 0$. (Hint: Don't forget the fact that $P(t) \in S$ by the definition of S).
    5. Show that $P(t) = 0$ if and only if $t \in S^\perp$. Thus, $S^\perp$ is the Eigen space corresponding to Eigen value 0.
    6. Show that 0 and 1 are the only Eigen values of P.