CS 6101 MFCS Test-I, Aug. 2018

1. Let A be an $n \times n$ real matrix satisfying $x^T Ax > 0$ for all $0 \neq x \in \mathbf{R}^n$ (i.e., A is positive definite). Show that A is non-singular. □

   Soln: It is sufficient prove that the columns of A are linearly independent and for that it is sufficient to prove that whenever $Ax = 0$ for any $x \in \mathbf{R}^n$ then $x = 0$ (why?). Suppose $Ax = 0$ then, $x^T Ax = 0$. By positive definiteness of A, this implies $x = 0$, proved.

2. Consider $\mathbf{R}^2$ with the standard inner product $()$. Let $u = [1, 1]^T$ and $v = [1, 0]^T$. Find a scalar value α such that $w = v - \alpha u$ satisfies $(u, w) = 0$. □

   Soln: First normalize u to a unit vector. This gives $u' = \frac{1}{\sqrt{2}}[1, 1]^T$. Now find $v - (v, u')u' = v - \frac{(v,u')}{||u||}u$ to get a vector perpendicular to v. Thus $\alpha = \frac{(v,u')}{||u||} = \frac{(v,\frac{u}{||u||})}{||u||} = \frac{(v,u)}{||u||^2} = \frac{(v,u)}{(u,u)}$. Calculations yield $\alpha = \frac{1}{2}$.

3. Let $()$ be an inner product on the vector space V. For any two vectors $u, v \in V$, we say u, v are orthogonal (with respect to the inner product $()$) if $(u, v) = 0$. Let b_1, b_2 be arbitrary non zero vectors in V. Find a scalar α such that $b_2 - \alpha b_1$ is orthogonal to b_1. □

   Soln: This is just the generalization of the previous question. We have $\alpha = (b_2, \frac{b_1}{||b_1||})\frac{b_1}{||b_1||} = \frac{(b_2,b_1)}{(b_1,b_1)}$. It is easy to verify that for this value of α, $(b_1, b_2 - \alpha b_1) = 0$.

4. Let $()$ be an inner product on the vector space V. Let $u, v \in V$ be non zero vectors satisfying $(u, v) = 0$. Show that u, v are linearly independent. □

   Soln: Suppose $\alpha u + \beta v = 0$. We have to prove that $\alpha = \beta = 0$. Now $\alpha u + \beta v = 0 \implies (u, \alpha u + \beta v) = 0 \implies \alpha(u, u) + \beta(u, v) = 0$. Now $(u, v) = 0$ because u, v are orthogonal. Hence $\alpha(u, u) = 0$. Since $u \neq 0$, we have $(u, u) > 0$ and hence $\alpha = 0$. Similarly $\beta = 0$.

5. Consider the vector space of polynomials of degree at most 4 with real coefficients. Consider the basis $1, (x - 2), (x - 2)^2, (x - 2)^3, (x - 2)^4$ of this vector space. Find the coordinates of the polynomial $p(x) = 1 + x + x^2 + x^3 + x^4$ with respect to this basis. □

   Soln: Let $p(x) = a_0 + a_1(x - 2) + a_2(x - 2)^2 + a_3(x - 3)^3 + a_4(x - 4)^4$. Setting $x = 2$ we get $a_0 = p(2) = 31$. Differentiating, we get $p'(x)|_{x=2} = 49 = a_1$. One more differentiation step yields $p''(x)|_{x=2} = 62 = 2a_2$. Thus $a_2 = 31$. Differentiating again, we get $p'''(x)|_{x=2} = 54 = 6a_3$. This gives $a_3 = 9$. Yet another differentiation yields $p''''(x)|_{x=2} = 24 = 24a_4$ yielding $a_4 = 1$.

6. For all $u, v \in \mathbf{R}^2$, define $(u, v) = u^T Av$ where $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Is (u, v) an inner product on $\mathbf{R}^2$? □

   Soln: The only non-trivial property to check is positivity. That is, to show that for all $[x, y]A[x, y]^T > 0$ whenever x, y are non-zero real numbers. But $[x, y]A[x, y]^T = 2x^2 + y^2 > 0$ if $x \neq 0 \neq y$.