MFCS September 2018

1. Find the inverse of the matrix $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. No credits for solving brute force! There is a simple way to solve, which you must explain. [3]

   Soln: The columns of the matrix are orthogonal vectors of Euclidean norm (length) 2 each. Thus $\frac{1}{2}H$ is orthogonal. Consequently $(\frac{1}{2}H)^{-1} = (\frac{1}{2}H)^T$. But as H is symmetric, $H^T = H$. Hence we have $(\frac{1}{2}H)^{-1} = \frac{1}{2}H$ or $H^{-1} = \frac{1}{4}H$

2. Let A be an $n \times n$ symmetric matrix. Let v, w be non-zero vectors in $\mathbf{R}^n$ such that $Av = \lambda_1 v$ and $Aw = \lambda_2 w$, where λ_1, λ_2 are scalars such that $\lambda_1 \neq \lambda_2$. Show that the standard inner product $(v, w) = 0$. (i.e., to prove that Eigen vectors corresponding to distinct Eigen values of a symmetric matrix are orthogonal. Hint: Consider the inner product of v with Aw). [3]

   Soln: First note that $(v, Aw) = v^T A w = v^T A^T w = (Av, w)$. (The second equality used the fact that A is symmetric.) Now $(Av, w) = (\lambda_1 v, w) = \lambda_1 (v, w)$ and $(v, Aw) = (v, \lambda_2 w) = \lambda_2 (v, w)$. Thus we have $(\lambda_1 - \lambda_2)(v, w) = 0$. As $\lambda_1 \neq \lambda_2$, we have $(v, w) = 0$.

3. Consider the vector $v = [1, 2, 3]^T$ in $\mathbf{R}^3$. Let P be the subspace spanned by the vectors $[1, 1, 0]^T$ and $[0, 1, 1]^T$ (essentially a plane). [3x3]

   1. Find vectors $u, w \in \mathbf{R}^3$ such that $v = u + w$ and u is a vector in the plane S and w is a vector orthogonal to S.
      Soln: Let $c_1 = [1, 1, 0]^T$ and $c_2 = [0, 1, 1]^T$. Normalize c_1 to yield unit vector $b_1 = \frac{1}{\sqrt{2}}[1, 1, 0]^T$ in P. Now, using Gram Schmidt process, $b_2 = \frac{c_2 - (c_2, b_1) b_1}{||c_2 - (c_2, b_1) b_1||} = \frac{1}{\sqrt{6}}[-1, 1, 2]^T$ is a unit vector orthogonal to b_1. Thus $[b_1, b_2]$ is an orthonormal basis for P.
      The orthogonal projection of v to P is given by $u = (v, b_1) b_1 + (v, b_2) b_2$. After calculations we get $u = [\frac{1}{3}, \frac{8}{3}, \frac{7}{3}]^T$. It is easy to see that $w = v - u = [\frac{2}{3}, \frac{-2}{3}, \frac{2}{3}]^T$ is orthogonal to u (why?)
   2. Find the distance v from the point in P nearest to v.
      Soln: The nearest point in P to v is indeed u. Thus, the distance of P from v is $d(v, u) = ||v - u|| = ||w|| = \frac{2}{\sqrt{3}}$.
   3. Find a 3×3 matrix A such that for any vector $v = [x, y, z]^T$, Av gives the component w of v perpendicular to the plane S.
      Soln: The projection matrix M_P defined by:
      $M_P = b_1 b_1^T + b_2 b_2^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 & 2 & -2 \\ 2 & 4 & 2 \\ -2 & 2 & 4 \end{bmatrix}$ has the property that $M_P(v)$ gives the orthogonal projection u of v along the plane P. Hence $w = v - M_P(v) = (I - M_P)(v)$, where I is the identify matrix. Thus the component w of v normal to u is obtained by multiplying v with the matrix

      $I - M_P = I - b_1 b_1^T + b_2 b_2^T = \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$,