Assignment I

Note: Many theory questions included here were discussed in the class. They are included here to refresh your memory. Some other questions are intended to give you practice with the theory. You do not have to submit the assignment. However, the examination question paper will be set assuming that you have acquired the knowledge and skill to solve the assignment questions.

1. Consider the space $\mathbf{C}^2$ with the standard (complex) inner product. Consider the following bases: $B = (b_1 = [1,1]^T, b_2 = [1,0]^T)$, $C = (c_1 = [0,1]^T, c_2 = [1,1]^T)$. Suppose vectors v_1 and v_2 have (complex) coordinates (z_1, z_2) and (y_1, y_2) respectively with respect to the basis B. (Note that coordinates are complex. For example, $1 = 1 + 0j$ etc.).
   1. Find the coordinates of v_1 with respect to the basis C.
   2. Find the inner product (v_1, v_2). (Note that neither B nor C is an orthonormal bases).
2. Let $[b_1, b_2, \dots b_n]$ and $[c_1, c_2, \dots c_n]$ be orthonormal basis of $\mathbf{C}^n$ such that $[b_1, b_2, \dots b_n] = [c_1, c_2, \dots c_n]Q$ for some unitary matrix Q. Let a vector $v \in \mathbf{C}^n$ have coordinates $[y_1, y_2, \dots y_n]$ w.r.t. basis $[c_1, c_2, \dots c_n]$. Show that the coordinates of v w.r.t. basis $[b_1, b_2, \dots b_n]$ will be will be $Q^* y$.
3. Let $|p\rangle = Z|0\rangle$ and $|q\rangle = Z|1\rangle$, where Z is the Pauli's Z gate. Will $(|p\rangle, |q\rangle)$ be an orthonormal basis of $\mathbf{C}^2$? Suppose $|v\rangle = \alpha|0\rangle + \beta|1\rangle$, find the coordinates of $|v\rangle$ w.r.t basis $(|p\rangle, |q\rangle)$.
4. Suppose an operator T in $\mathbf{C}^2$ satisfies: $T[1,1]^T = [1,0]^T$ and $T[1,-1]^T = [0,1]^T$
   1. Find the matrix of the operator with respect to the standard basis, $(|0\rangle, |1\rangle)$.
   2. Find the matrix of the operator with respect to the basis $([0,j]^T, [-j,0])$
5. Let T be a linear transformation in $\mathbf{C}^n$. Let $[b_1, b_2, \dots b_n]$ and $[c_1, c_2, \dots c_n]$ be orthonormal basis of $\mathbf{C}^n$ such that $[b_1, b_2, \dots b_n] = [c_1, c_2, \dots c_n]Q$ for some unitary matrix Q. Suppose the matrix of the linear transformation T with respect to the basis $[b_1, b_2, \dots b_n]$ is A, show that the matrix of T with respect to the basis $[c_1, c_2, \dots c_n]$ will be $QAQ^{-1} = QAQ^*$. (Convince yourself that if A and Q are unitary, QAQ^* is also unitary, as one would expect).
6. Let A be a unitary $n \times n$ matrix. Let $[b_1, b_2, \dots b_n]$ and $[c_1, c_2, \dots c_n]$ be orthonormal basis of $\mathbf{C}^n$. Show that $[Ab_1, Ab_2, \dots Ab_n]$ also is an orthonormal basis of $\mathbf{C}^n$.
7. Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - |1\rangle$. What is the matrix of the Pauli X, Y and Z gates with respect to the basis $(|+\rangle, |-\rangle)$?
8. For all $x, y \in \mathbf{C}^n$, show that $(x, y) = \frac{1}{4}(||x+y|| + ||x-y|| - j||x - jy|| + j||x + jy||)$. Using this equality, prove that in $\mathbf{C}^n$, if a linear transformation T satisfies $||T(z)|| = ||z||$ for all $z \in \mathbf{C}^n$, then T is a unitary transformation.
9. Prove the Cauchy Schwartz inequality, $|(x, y)| \leq ||x||.||y||$ for all $x, y \in \mathbf{C}^n$.
10. If A and B are unitary transformations in $\mathbf{C}^n$ then show that AB is also unitary.
11. Show that $X = HZH$ where X, Z are the Pauli gates and H the Hadamard gate.
12. If $x, a \in \mathbf{C}^n$ and $y, b \in \mathbf{C}^m$. Let A and B be unitary operators in $\mathbf{C}^n$ and $y, b \in \mathbf{C}^m$ respectively.
    1. Show that $(x \otimes y, a \otimes b) = (x, a)(y, b)$.
    2. Show that $(A \otimes B)(x \otimes y) = Ax \otimes By$.
    3. Show that if $[b_1, b_2, \dots, b_n]$ and $[c_1, c_2, \dots c_m]$ are orthonormal basis of $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively, $[b_i \otimes c_j]_{1 \leq i \leq n, 1 \leq j \leq m}$ is an orthonormal basis for $C^n \otimes C^m$.

13. Let A, B be unitary transformations in $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively. Let $[b_1, b_2, \ldots, b_n]$ and $[c_1, c_2, \ldots c_m]$ be orthonormal basis for $\mathbf{C}^n$ and $\mathbf{C}^m$ respectively. Show that $[Ab_i \otimes Bc_j]_{1 \leq i \leq n, 1 \leq j \leq m}$ is an orthonormal basis for $C^n \otimes C^m$.

14. Write down the coordinates with respect to the standard basis $[|00\rangle, |01\rangle|10\rangle|11\rangle]$ of the vectors $\mathbf{C}^4$ forming the basis $[X|r\rangle \otimes Z|s\rangle]_{r,s \in \{|0\rangle,|1\rangle\}}$, where X and Y are the Pauli matrices.

15. Find the matrix representation of the operator $X \otimes Z$ with respect to the standard basis $[|00\rangle, |01\rangle|10\rangle|11\rangle]$.

16. A quantum gate in $\mathbf{C}^4$ is specified by $T(|r\rangle \otimes |s\rangle) = H|r\rangle \otimes |s\rangle$ for $r, s \in \{|0\rangle, |1\rangle\}$, where H is the 2×2 Hadamard matrix. Find the 4×4 unitary matrix for T with respect to the standard basis $[|00\rangle, |01\rangle|10\rangle|11\rangle]$. (What happens if the specification is changed to $T(|r\rangle \otimes |s\rangle) = |r\rangle \otimes H|s\rangle$ for $r, s \in \{|0\rangle, |1\rangle\}$?)

17. Let A be any $n \times n$ unitary matrix. Let $\mathbf{0}$ be the $n \times 1$ column vector of n zeroes. A controlled A gate is defined by $C_A = \begin{bmatrix} 1 & \mathbf{0^T} \\ \mathbf{0} & A \end{bmatrix}$. Show that C_A is unitary when A is unitary.

18. Write down the 4×4 matrix w.r.t. the standard basis $[|00\rangle, |01\rangle|10\rangle|11\rangle]$ for

    1. Controlled Z gate, C_Z w.r.t. the standard basis
    2. Controlled NOT gate, C_X, where X is the Pauli X matrix.

19. Consider the quantum gate in $\mathbf{C}^4$ whose matrix w.r.t the standard basis is $A = (I \otimes H)C_Z(I \otimes H)$. What is the connection of A to the C_X gate? This shows how to construct the controlled not gate using Pauli X gate and Hadamard gate.

20. Consider the quantum gate G specified by $G(|r\rangle \otimes |s\rangle) = C_X(H|r\rangle \otimes |s\rangle)$. Find the representation of the quantum states $\beta_{00} = G|00\rangle$, $\beta_{01} = G|01\rangle$, $\beta_{10} = G|10\rangle$ and $\beta_{11} = G|00\rangle$ with respect to the standard basis. These states are called *Bell states*.