## Assignment II

1. Given a graph $G(V, E)$, we have seen how to formulate the minimum vertex cover problem (MINVC) as a hitting set problem. Instead, recall that we can formulate the problem directly as an LP and write down the dual. The LP formulation was Min: $\sum_{v \in V} x_v$ subject to: $x_u + x_v \geq 1$ for each $(u, v) \in E(G)$. The dual is the maximum matching problem (MAXMATCH).

   Consider the primal dual algorithm applied to this problem:

   1. Intialize all primal and dual variables to $0$
   2. While there is an unsatisfied primal constraint
      - raise the value of the corresponding dual variable till some dual constraint becomes tight.
      - Set both the primal variables corresponding to tight dual constraint to $1$.
   3. Return the primal solution corresponding to primal variables having value $1$.

   Aruge that this algorithm is equivalent to the following:

   - Repeat
     - pick an edge in the graph and pick both its end points in the vertex cover.
     - Remove the two vertices (and all edges out of them) from the graph.
   - Until the graph has no more edges.

   What is the approximation ratio achieved by the algorithm? Is it possible to get a dual feasible solution and hence a matching from the algorithm? In this case, what can you say about the approximation ratio achieved by the dual solution?

2. In this question we will show that the vertices of the feasible polytope (equivalently extreme points or basic feasible solutions - as stated in the class) for MINVC has the interesting property that values of the variables can only be either $0$, $1$ or $\frac{1}{2}$. This is called the half integrality property of MINVC.

   Suppose $(x_v)$, $v \in V(G)$ be any feasible solution to MINVC. Suppose this solution does not have the half integrality property, we will show that this point can be expressed a convex combination of two other feasible points, and hence cannot be an extreme point (and hence a vertex) of the polytope.

   Consider the set $V^+ = \{x_v\colon \frac{1}{2} < x_v < 1\}$ and $V^- = \{x_v\colon 0 < x_v < \frac{1}{2}\}$. For any $\epsilon > 0$, consider the following set of variables: $y_v = x_v + \epsilon$ if $x_v \in V^+$, $y_v = x_v - \epsilon$ if $x_v \in V^-$, $y_v = x_v$ otherwise; and $z_v = x_v - \epsilon$ if $x_v \in V^+$, $z_v = x_v + \epsilon$ if $x_v \in V^-$, $z_v = x_v$ otherwise.

   1. Show that if $(y_v)$ and $(z_v)$ are feasible for any $\epsilon > 0$ then $(x_v)$ cannot be an extreme point of the polytope.
   2. Show that if $(x_v)$ is not half integral, then a value for $\epsilon > 0$ can be always found out such that both $(y_v)$ and $(z_v)$ are feasible.

3. Consider an $n \times n$ chess board. Let $x_{i,j}$ denote position $(i, j)$ for $1 \leq i, j \leq n$. Each position has an associated cost function $c_{i,j}$. Consider the problem of placing **rooks** on the chess board such that the rooks do not attack each other and the cost of the positions where rooks are placed adds up to the **minimum** possible.

   - Formulate the problem as an $LP$ and find the dual formulation. Show that you can replace the equality constrains in the formulation to inequality constraints with no loss of generality to get a formulation in standard form. Thus the constraints become $\sum_j x_{ij} \geq 1$ for each $i$ and $\sum_i x_{ij} \geq 1$ for each $j$.
   - Suppose the dual variables are $y_i$ and $z_i$ for $1 \leq i \leq n$. Suppose $(x_{ij})$ and $(y_i), (z_i)$ are primal and dual feasible values. Consider the set $E = \{(i, j) : y_i + z_j = c_{ij}\}$. $E$ is essentially the set of "tight" $(i, j)$ pairs. Suppose we can find a subset $E' \subseteq E$ such that $|E'| = n$, then show that $E'$ must be an optimal feasible solution to the problem.