Name and Roll No.: ______________________________

1. Given sets $S_1, S_2, .., S_n$ that are subsets of a set S of m elements, and weights $c_1, c_2, .., c_n$ for the sets, the SET COVER problem (SC) asks you to pick a subcollection of $\{S_1, S_2, ..., S_n\}$ of as small total weight as possible so that every element in S appears in at least one of the sets in the subcollection that you have picked. You are also told that an element can appear in at most t sets. 3x3

   - Formulate the problem as a $0-1$ LP and Write down the dual.
     Soln: Let $s_1, s_2, ..., s_m$ be the elements of S. To objective is to:
     Minimize $\sum_{i=1}^{n} c_j x_j$ subject to $\sum_{s_i \in S_j} x_j \geq 1$ for each $1 \leq i \leq m$, $x_j \in \{0,1\}$.
     The dual is to: Maximize $\sum_{i=1}^{n} y_i$ subject to $\sum_{s_i \in S_j} y_i \leq c_j$, $y_i \in \{0,1\}$.
   - Suppose you solve the LP relaxiation optimally, What must be the rounding criteria to set a variable to value 1 for getting a t factor approximation? Justify.
     Soln: Each constraint of the primal has at most t primal variables. Hence, in any feasible solution to the primal, at least one of the variables must have value greater than or equal to $\frac{1}{t}$ (why?). If we pick the sets corresponding to all variables with value $\frac{1}{t}$ or more in the optimal solution to the LP relaxiation, we will have a feasible solution (why?). Moreover, since every variable gets multiplied by atmost t, this must lead to a t factor approximation (why?).
   - What is the factor of approximation achieved by the following primal dual algorithm? Justify.
     - Repeat: Increase the value of a dual variable corresponding to any unsatisfied primal constraint till some dual constraint becomes tight and set all primal variables corresponding to tight dual constraints to 1
     - Until: There are no more unsatisfied primal constrains.

   Soln: Here, since each primal constraint has at most t variables, in any feasible solution, the sum of the variables in a primal constraint can be at most t. The dual constraints are tight by design (of the algorithm). Thus, the primal constraints are relaxed by a factor of at most t while dual constraints remain tight. We have seen in the class that when this is the case, the solution that results must be a factor t approximation.

2. The CONSTRAINT SATISFACTION PROBLEM (CSP) takes as input a set of equations over n boolean variables with only $\oplus$ operation permitted. (Ex: $\{x_1 \oplus x_2 = 1, x_2 \oplus x_3 = 0, x_3 \oplus x_1 = 0\}$ - essentially a set of linear constraints over $GF(2)$). The problem is to find a truth assignment to the boolean variables that maximizes the number of satified constraints. 3+3

   - What is the expected number of constraints satisfied by a random truth assignment to the variables? Justify.
     Soln: Given a formula of the form $y_1 \oplus y_2 \oplus ... \oplus y_n$ in n boolean variables, it is not hard to see that exactly half of the 2^n truth assignments to the variables make the evaluate to 1. (To see this in a more structured way, suppose we fix the values of the first $n-1$ variables. Then the probability of that a random truth assignment to the last variable satisfies the formula is the probability that the truth value to this variable differs from the XOR of all the fixed variables and is $\frac{1}{2}$ (why?)). Since each constraint is satisfied with probability $\frac{1}{2}$, the expected number of satisfied constraints must be half the total number.
   - How can you de-randomize the random truth assignment? What is the expected number of satisfied constrains under your derandomization strategy?
     Soln: Set the first variable x_1 to zero and we get a new constraint system. If we set x_1 to one, we get another constriant system. In both cases, we know that random truth assignment to the remaining variables satisfy half the number of constraints in expectation. Since the expectation of the orginal CSP instance is the average of these two values, at least one of them must be greater than or equal to half the total number of constrains in the orignal CSP. The value of x_1 can be fixed to get the larger expectation and we continue this way to get a de-randomized factor two approximation.

3. Consider an LP of the form Max: $< c, x >$ subject to: $Ax \leq b, x \geq 0$, where $c, x \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$. $\boxed{3+3}$

   - Show that if the LP has more than one optimal solutions, then it has infinitely many optimal solutions.
     Soln: If x_1, x_2 are feasible solutions with $c^T x_1 = c^T x_2 = t$, then $c^T(\lambda x_1 + (1-\lambda)x_2) = t$ as well for any value of λ. Since, the feasible region is **convex**, for values of λ in $0 \leq \lambda \leq 1$, $x = \lambda x_1 + (1-\lambda)x_2$ must be feasible. If x_1, x_2 are optimal, so are all these points.
   - Suppose there exists a $y \in \mathbf{R}^m$ such that $A^T y = 0_n$ and $y \geq 0_m$, then the LP does not have any feasible solution. (Prove from first principles, do not use the theorem of alternatives).
     Soln: The question has an error. $b^T y \geq (Ax)^T y = x^T A^T y = x^T(A^T y) = x^T 0_n = 0_n$. A Contradiction would have arisen only if the condition $b^T y < 0$ was also given in the question!

4. For the Max2Sat instance $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2)$, $\boxed{3\text{x}3}$

   - Write down the quadratic program.
     Soln: Using the QP variables y_0, y_1, y_2 taking values in $\{1, -1\}$, we can encode each clause using a quadratic formula. Consider $(x_1 \vee \neg x_2) = \neg(\neg x_1 \wedge x_2)$. we use the encoding $f_1(y_0, y_1, y_2) = 1 - \frac{(1-y_0y_1)}{2}\frac{(1+y_0y_2)}{2}$ for the first clause. Similarly, the encoding $f_2(y_0, y_1, y_2) = 1 - \frac{(1+y_0y_1)}{2}\frac{(1-y_0y_2)}{2}$ captures the second clause. For $j \in \{1, 2\}$ x_j evaluates to 0 if $y_0 = y_j$ and x_j evaluates to 1 if $y_0 \neq y_j$. With this, it is easy to see that the optimal solution to (y_0, y_1, y_2) in $\{-1, 1\}^3$ maximizing $f_1 + f_2$ also maximizes the number of clauses satisfied in the orignal formula and hence the quadratic program captures the original problem. Note that $f_1 + f_2$ simplifies to $\frac{(3+y_1y_2)}{2}$. Also note that maximizing this objective function is equivalent to maximizing y_1y_2.
   - Write down the vector program. Find an optimal solution which does not contain any of the standard basis vectors $(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T$.
     Soln: The vector program optimizes the above quadratic optimization problem by allowing the relaxed constraints $y_0, y_1, y_2 \in \mathbf{R}^3$, and with multiplication replaced by dot product of vectors. The additional norm constraint that $||y_i|| = 1$ is also imposed. Thus, to maximize y_1y_2, any feasible solution that sets $y_1 = y_2$ suffices. In particular $y_0 = y_1 = y_2 = \frac{(1,1,1)}{\sqrt{3}}$ is a solution. (This choice makes answering the next question easiest!)
   - Suppose the random vector $\frac{1}{\sqrt{3}}(1, 1, 1)$ is chosen for random hyperplane approximation, What is the solution obtained to the quadratic program for the Max2Sat Instance?
     Soln: Clearly $r^T y_0 = r^T y_1 = r^T y_2 = 1$. As all dot products are positive, we set $y_0 = y_1 = y_2 = 1$ as the solution for the vector program and correspondingly, we get $x_1 = x_2 = 1$ for the Max2Sat instance. Note that the objective function achieves the maximum (satisfying all clauses) with this assignment.