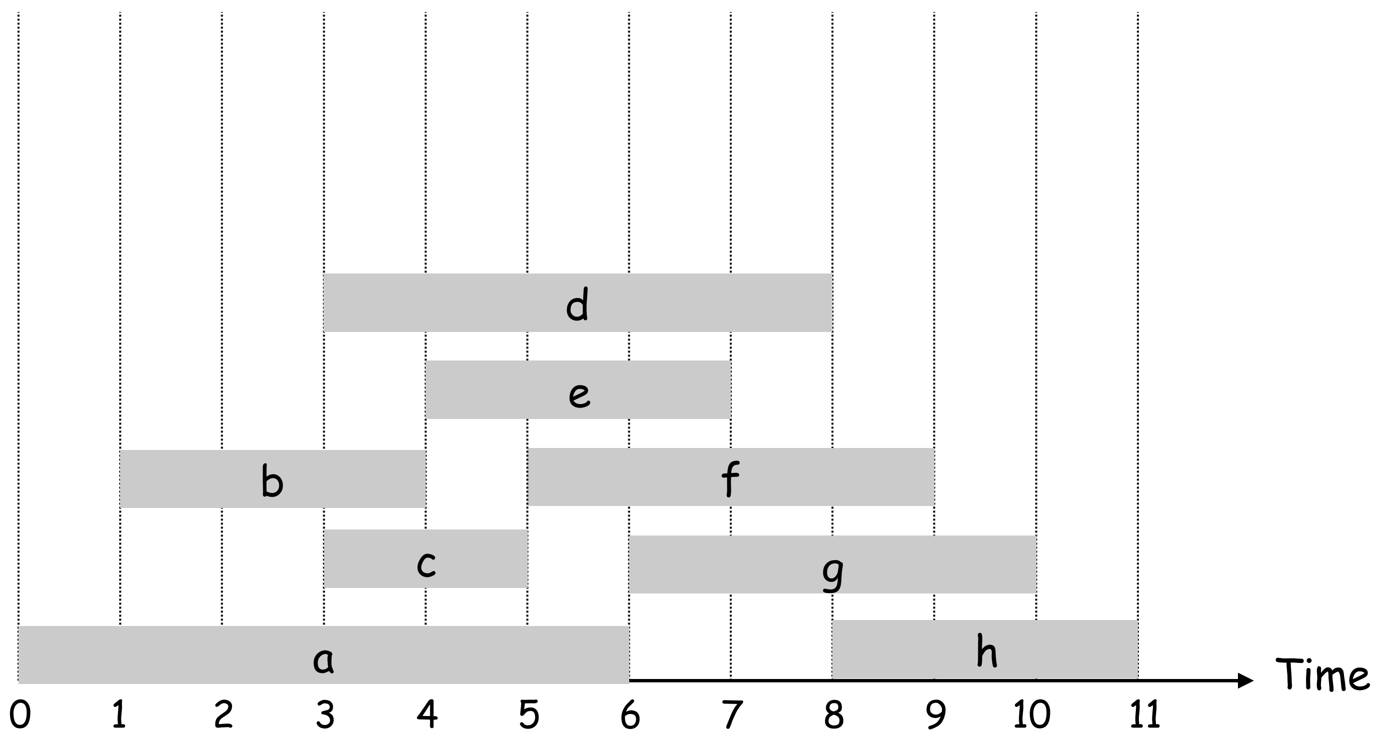
Interval Scheduling: Greedy Algorithms and Dynamic Programming



Slides based on Kevin Wayne / Pearson-Addison Wesley

Overview of Interval Scheduling

The Basic Interval Scheduling Problem

- Schedule as many non-overlapping tasks as possible in given timeframe
- (Representative problem #1 from day #1)

Total Interval Scheduling

- Must schedule all tasks
- Identify the fewest number of processors needed to schedule within given timeframe

Weighted Interval Scheduling

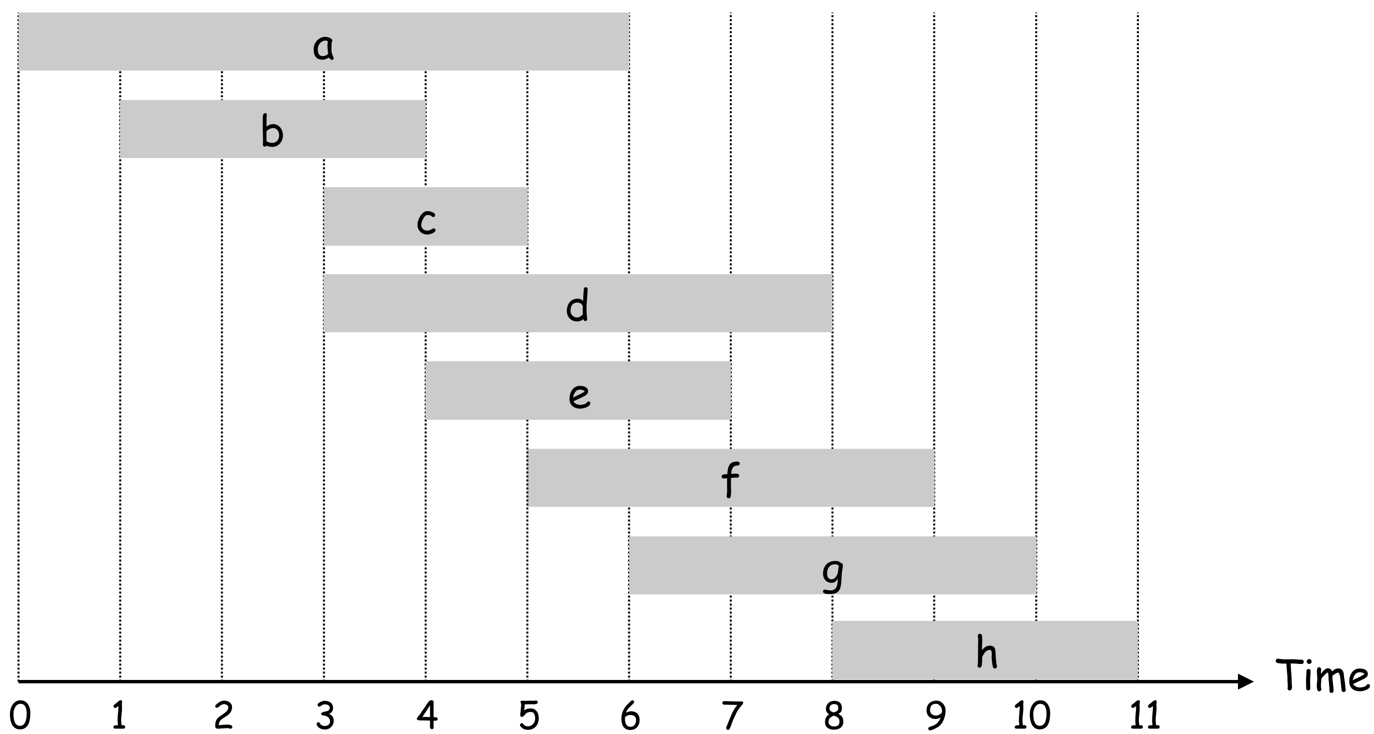
- Schedule non-overlapping tasks of maximum weight in given timeframe
- (Representative problem #2 from day #1)

We'll look for greedy solutions when possible, and use dynamic programming when greedy algorithms don't appear to work out.

Interval Scheduling

Interval scheduling.

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of start time s_j.
- [Earliest finish time] Consider jobs in ascending order of finish time f_j.
- [Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.
- [Fewest conflicts] For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

breaks earliest start time

breaks shortest interval

breaks fewest conflicts

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
INTERVAL-SCHEDULING( s1, f1, …, sn, fn )
1. Remain = {1,…,n}
2. Selected = {}
3. while ( |Remain| > 0 ) {
4.     k = argmin i ∈ Remain fi
5.     Selected = Selected ∪ {k}
6.     Remain = Remain - {k}
7.     for every i in Remain {
8.         if (si < fk) then Remain = Remain - {i}
9.     }
10. }
11. RETURN Selected
```

Implementation. $O(n^2)$.

- While loop is $O(n)$.
- Inside of loop is $O(n)$. (Argmin is $O(n)$. Updating Remain is $O(n)$.)

Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

 ↙ jobs selected
A ← φ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. O(n log n).

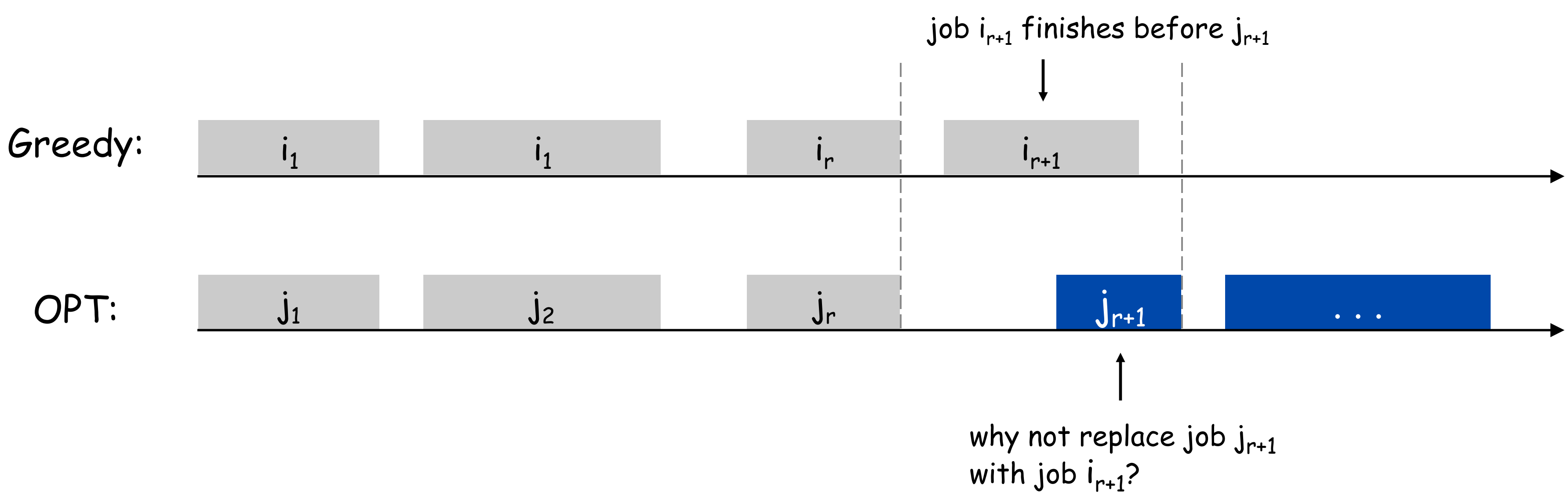
- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.

Interval Scheduling: Analysis

Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



Interval Scheduling: Analysis

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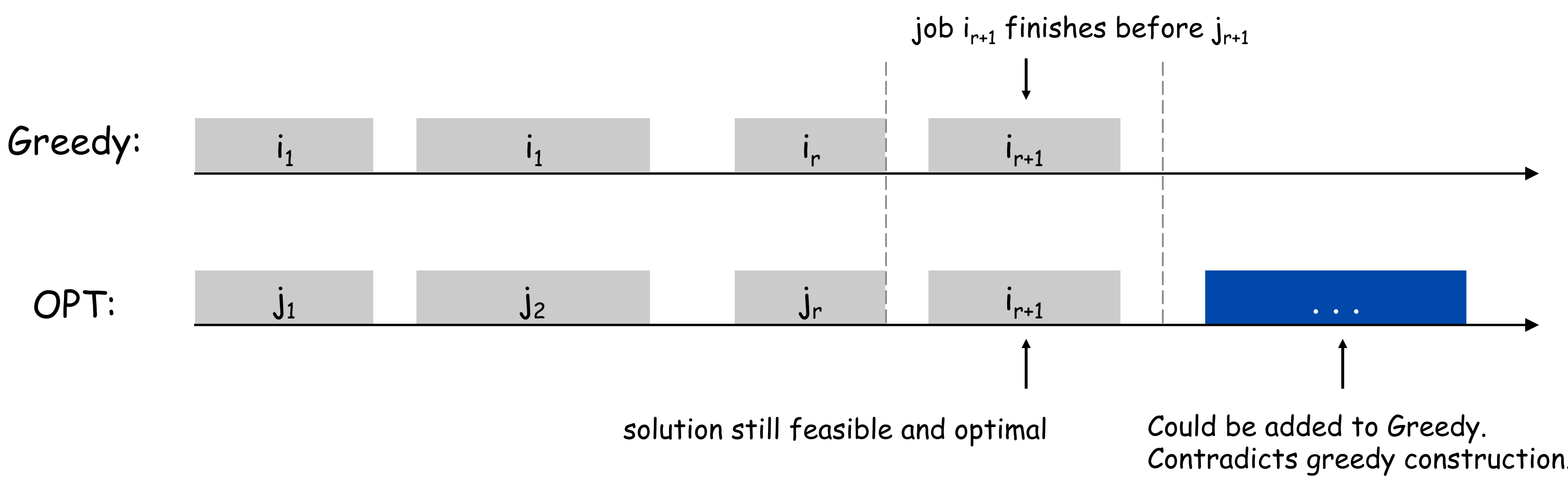


Interval Scheduling: Analysis

Interval Scheduling by Dynamic Programming

Could this problem also be solved by dynamic programming?

- Yes. Sort by finish time.
- Let S[k] = max(S[k-1], 1 + S[j])
  - Where k is the items (intervals) ordered by finish time
  - Where $j < k$ is the largest index such that the finish time of item j does not overlap the start time of item k

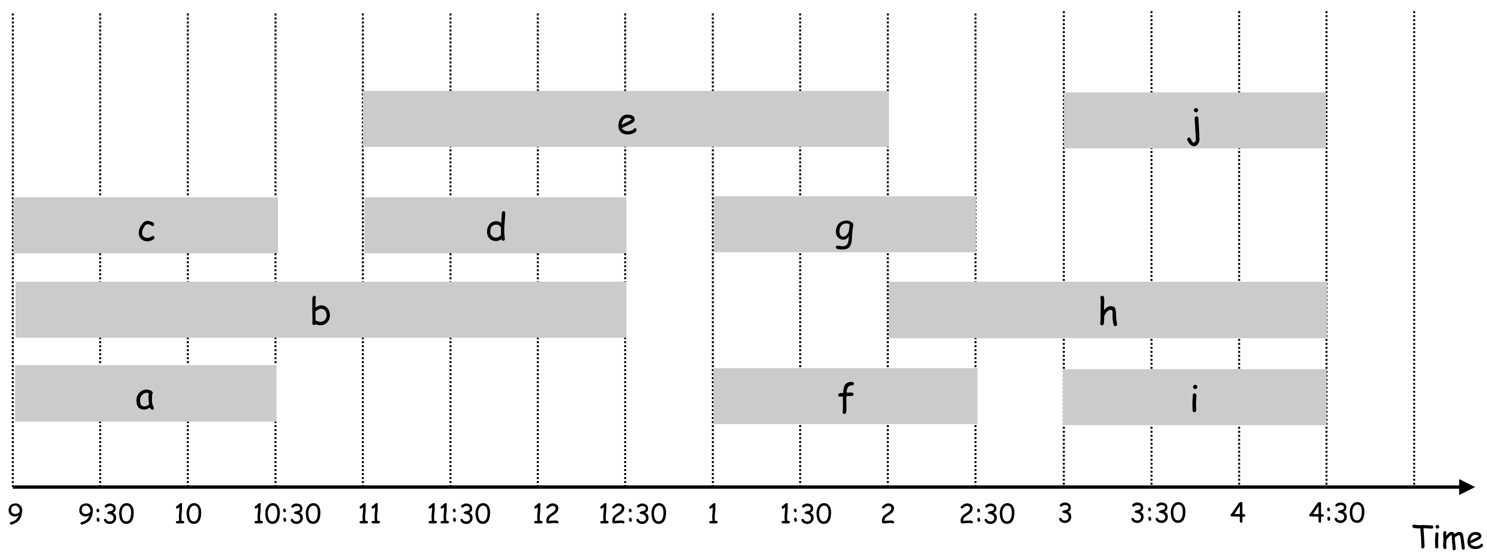
Interval Partitioning: Scheduling All

Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.

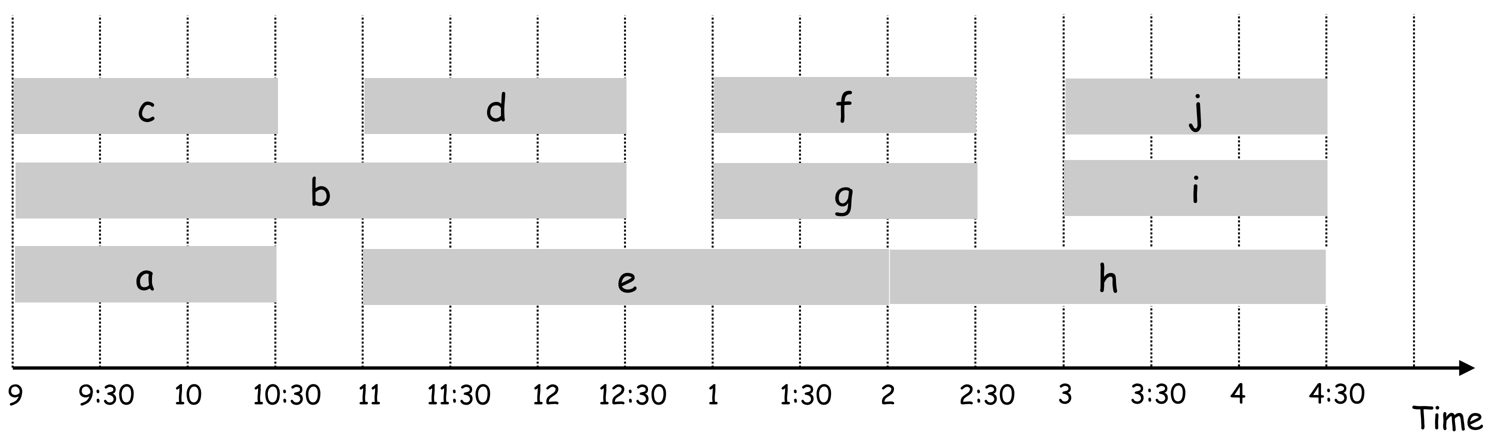


Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: Lower Bound on Optimal Solution

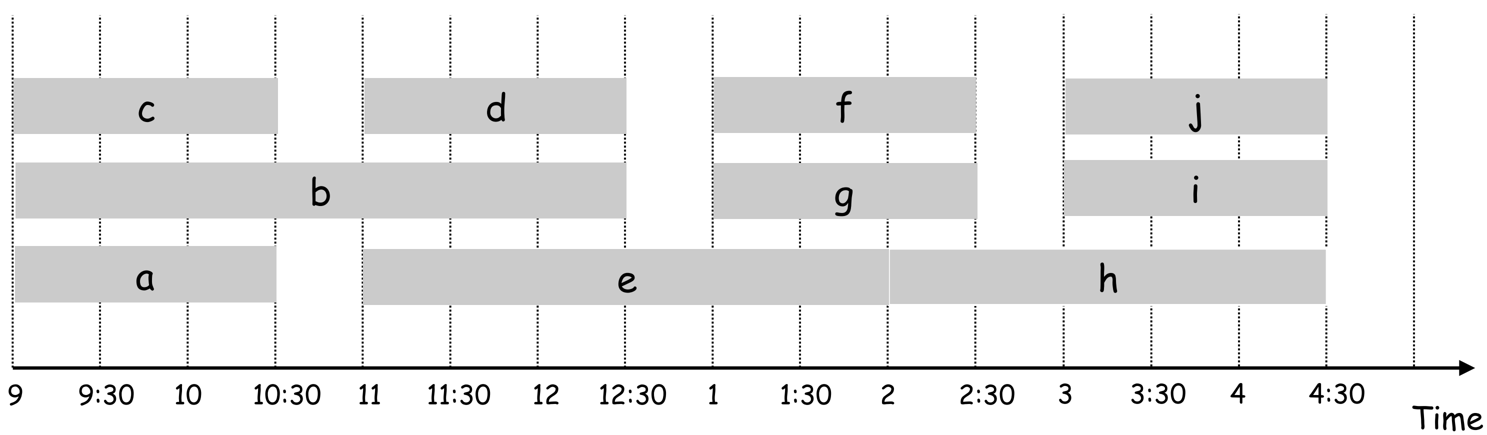
Def. The depth of a set of open intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0    ← number of allocated classrooms

for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation. $O(n \log n)$.

- For each classroom k, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

Interval Partitioning: Greedy Analysis

Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

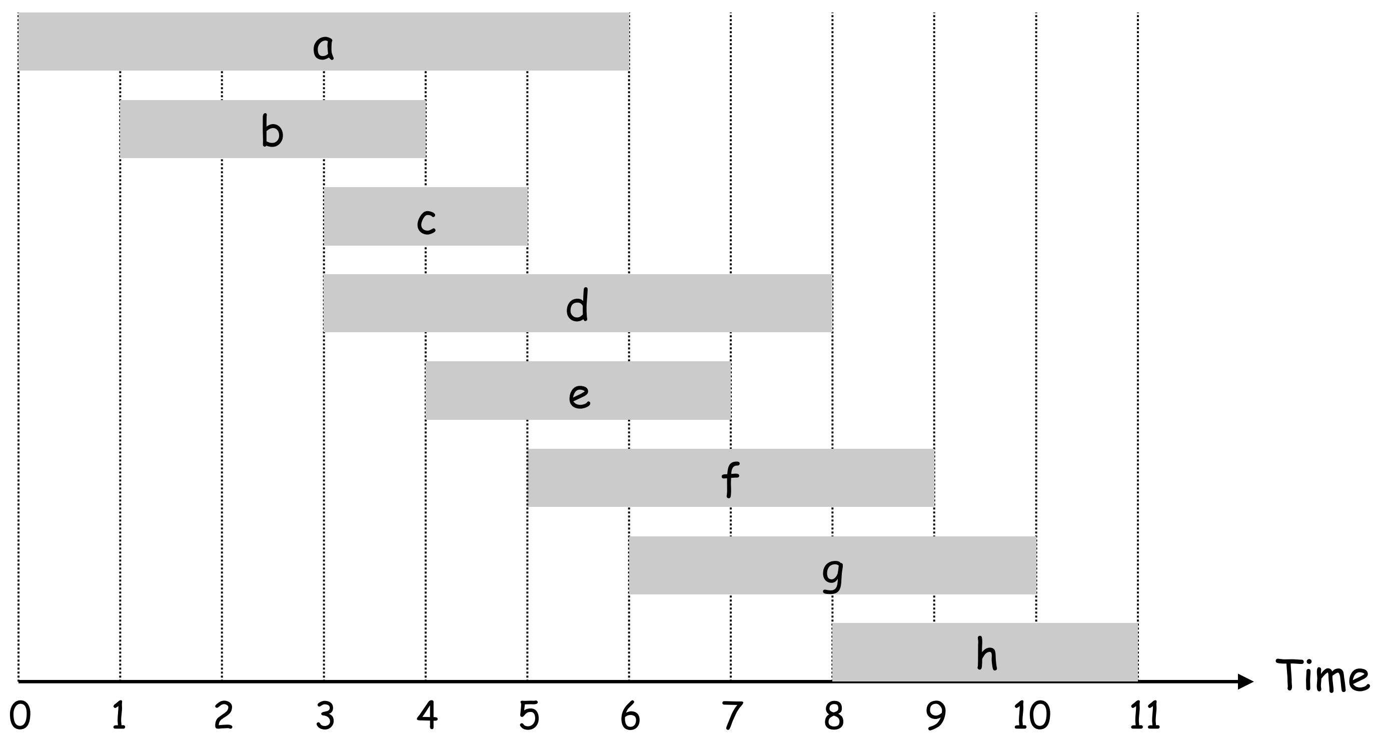
- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation $\Rightarrow$ all schedules use $\geq$ d classrooms. ▪

Weighted Interval Scheduling

Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job j starts at s_j, finishes at f_j, and has weight/cost/value v_j .
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

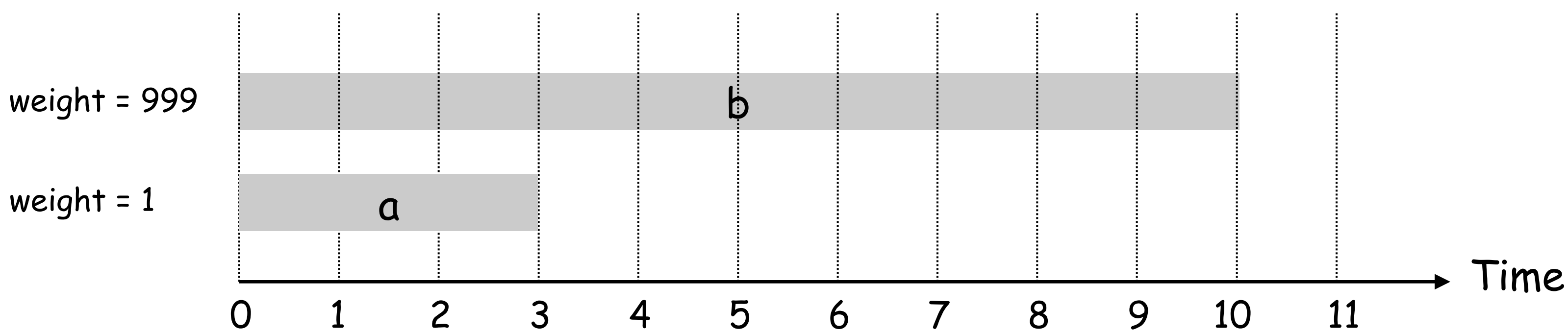


Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

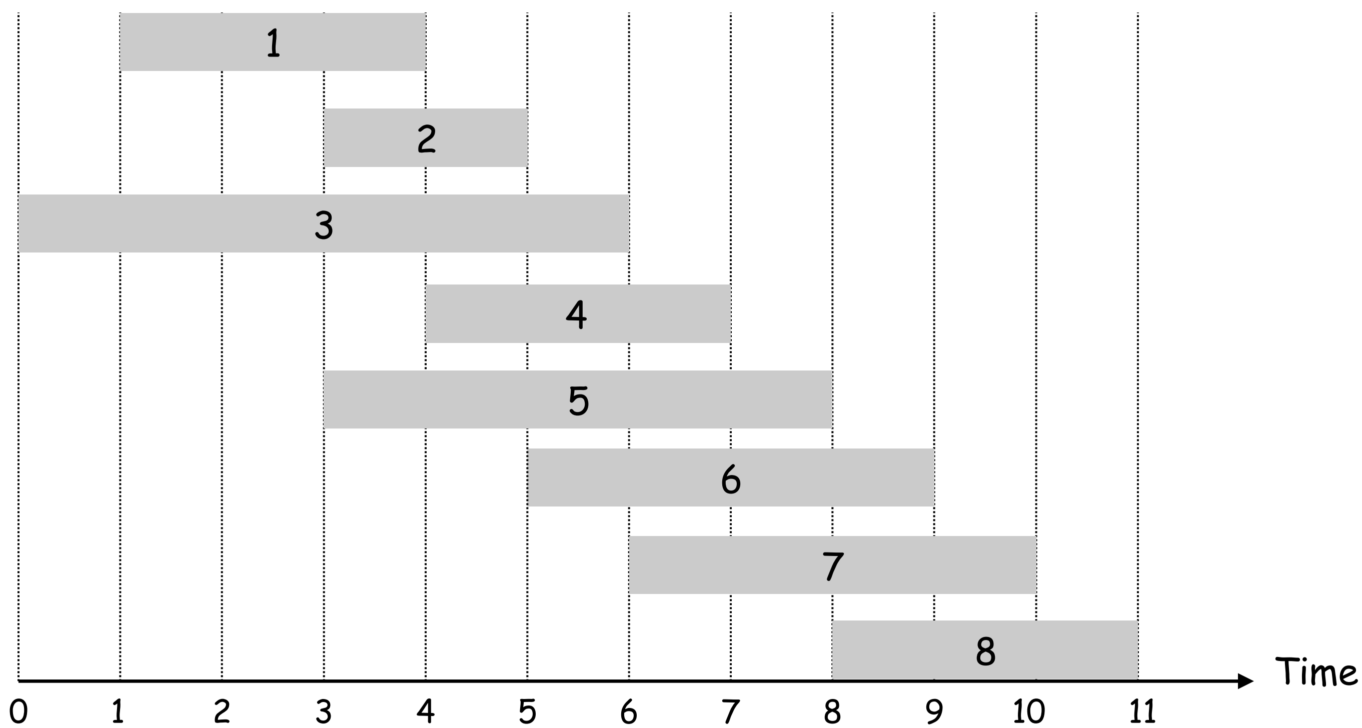
Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.



Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$.
Def. p(j) = largest index i < j such that job i is compatible with j.

Ex: p(8) = 5, p(7) = 3, p(2) = 0.



Dynamic Programming: Binary Choice

Notation. S[j] = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

- Case 1: j is selected.
  - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)

optimal substructure

- Case 2: j is not selected.
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$S[j] = \begin{cases} 0 & \text{if } j = 0 \\ \max \left\{ v_j + S[p(j)],\ S[j-1] \right\} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

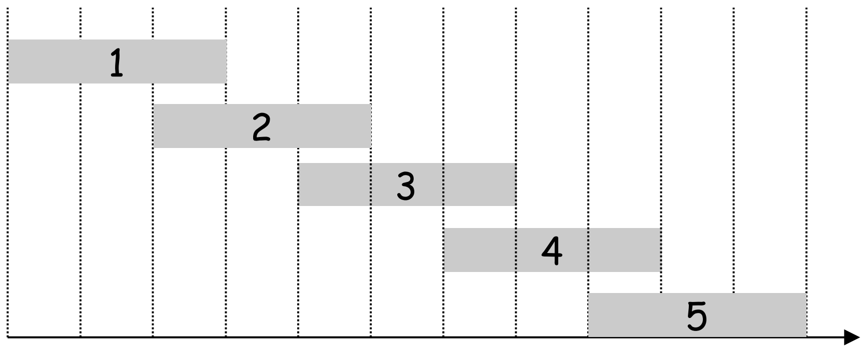
Brute force algorithm.

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max(vj + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

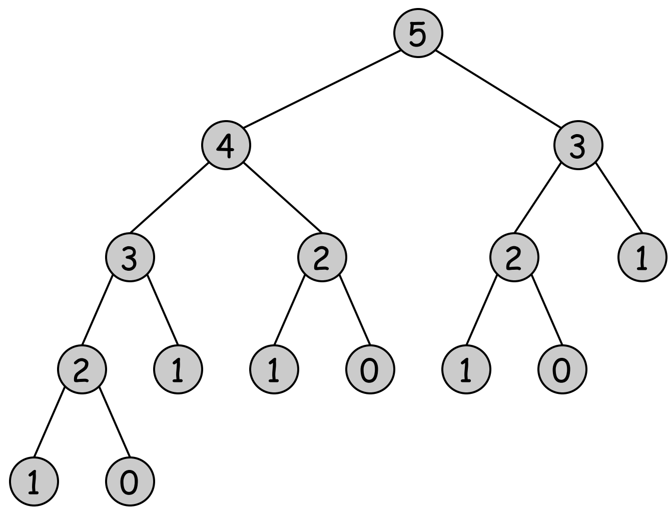
Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems $\Rightarrow$ exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



$p(1) = 0, p(j) = j-2$



Improved Complexity

Top-down dynamic programming: Memoization.

Bottom-up dynamic programming. Unwind recursion.

Running Time. $O(n \log n)$ to sort. $O(n^2)$ for straight forward computation of all p(i). (Can be done in $O(n \log n)$ by also sorting jobs by start time.) $O(n)$ for iterative loop.

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   S[0] = 0
   for j = 1 to n
      S[j] = max(v_j + S[p(j)], S[j-1])
}
```