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| **CS 598CSC: Approximation Algorithms** | **Lecture date: 1/28/11** |
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Notes edited by instructor in 2011.

We introduce the use linear programming (LP) in the design and analysis of approximation algorithms.The topics include Vertex Cover, Set Cover, randomized rounding, dual-fitting. It is assumed that the students have some background knowledge in basics of linear programming.

# 1 Vertex Cover via LP

Let $G = (V, E)$ be an undirected graph with arc weights $w : V \to R^+$. Recall the vertex cover problem from previous lecture. We can formulate it as an integer linear programming problem as follows. For each vertex $v$ we have a variable $x_v$. We interpret the variable as follows: if $x_v = 1$ if $v$ is chosen to be included in a vertex cover, otherwise $x_v = 0$. With this interprtation we can easily see that the minimum weight vertex cover can be formulated as the following integer linear program.

$$\begin{aligned}
\min \quad & \sum_{v \in V} w_v x_v \\
\text{subject to} \quad & \\
x_u + x_v &\geq 1 \qquad \forall e = (u, v) \in E \\
x_v &\in \{0, 1\} \qquad \forall v \in V
\end{aligned}$$

However, solving integer linear programs is NP-Hard. Therefore we use Linear Programming (LP) to approximate the optimal solution, $\mathrm{OPT}(I)$, for the integer program. First, we can relax the constraint $x_v \in \{0, 1\}$ to $x_v \in [0, 1]$. It can be further simplified to $x_v \geq 0$, $\forall v \in V$.

Thus, a linear programming formulation for Vertex Cover is:

$$\begin{aligned}
\min \quad & \sum_{v \in V} w_v x_v \\
\text{subject to} \quad & \\
x_u + x_v &\geq 1 \qquad \forall e = (u, v) \in E \\
x_v &\geq 0
\end{aligned}$$

We now use the following algorithm:

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| <u>Vertex Cover via LP</u>: |
| Solve LP to obtain an optimal fractional solution $x^*$ |
| Let $S = \{v \mid x^*_v \geq \frac{1}{2}\}$ |
| Output $S$ |

Then the following claims are true:

**Claim 1** *$S$ is a vertex cover.*

**Proof:** Consider any edge, $e = (u, v)$. By feasibility of $x^*$, $x_u^* + x_v^* \geq 1$, and thus either $x_u^* \geq \frac{1}{2}$ or $x_v^* \geq \frac{1}{2}$. Therefore, at least one of $u$ and $v$ will be in $S$. $\square$

**Claim 2** *$w(S) \leq 2\mathrm{OPT}_{LP}(I)$.*

**Proof:** $\mathrm{OPT}_{LP}(I) = \sum_v w_v x_v^* \geq \frac{1}{2} \sum_{v \in S} w_v = \frac{1}{2} w(S)$ $\square$

Therefore, $\mathrm{OPT}_{LP}(I) \geq \frac{\mathrm{OPT}(I)}{2}$ for all instances $I$.

**Note:** For minimization problems: $\mathrm{OPT}_{LP}(I) \leq \mathrm{OPT}(I)$, where $\mathrm{OPT}_{LP}(I)$ is the optimal solution found by LP; for maximization problems, $\mathrm{OPT}_{LP}(I) \geq \mathrm{OPT}(I)$.

## Integrality Gap

We introduce the notion of *integrality gap* to show the best approximation guarantee we can acquire by using the LP optimum as a lower bound.

**Definition:** For a minimization problem $\Pi$, the integrality gap for a linear programming relaxation/formulation $LP$ for $\Pi$ is $\sup_{I \in \pi} \frac{\mathrm{OPT}(I)}{\mathrm{OPT}_{LP}(I)}$.

That is, the integrality gap is the worst case ratio, over all instances $I$ of $\Pi$, of the integral optimal value and the fractional optimal value. Note that different linear programming formulations for the same problem may have different integrality gaps.

Claims 1 and 2 show that the integrality gap of the Vertex Cover LP formulation above is at most 2.

**Question:** Is this bound tight for the Vertex Cover LP?

Consider the following example: Take a complete graph, $K_n$, with n vertices, and each vertex has $w_v = 1$. It is clear that we have to choose $n - 1$ vertices to cover all the edges. Thus, $\mathrm{OPT}(K_n) = n - 1$. However, $x_v = \frac{1}{2}$ for each $v$ is a feasible solution to the LP, which has a total weight of $\frac{n}{2}$. So gap is $2 - \frac{1}{n}$, which tends to 2 as $n \to \infty$.

## Other Results on Vertex Cover

1. The current best approximation ratio for Vertex Cover is $2 - \Theta(\frac{1}{\sqrt{\log n}})$ [1].

2. Open problem: obtain a $2 - \varepsilon$ approximation or to prove that it is NP-hard to obtain $2 - \varepsilon$ for any fixed $\varepsilon > 0$. Current best hardness of approximation: unless P=NP, there is no 1.36 approximation for Vertex Cover [2].

3. The vertex cover problem can be solved optimally in polynomial time for bipartite graphs. This follows from what is known as K'onig's theorem.

4. The vertex cover problem admits a polynomial time approximation scheme (PTAS), that is a $(1 + \epsilon)$-approximation for any fixed $\epsilon > 0$, for planar graphs. This follows from a general approach due to Baker [**?**].