**CS373T** **October, 2007**
**Computational Complexity** **Exercise 2**

1. Show that NTIME($f(n)$)$\subseteq$ATIME($f(n)$). □
2. Show that **BPP**$\subseteq$ **PSPACE** □
3. Suppose a complexity class $\mathbf{C}$ has the property that whenever a language $L \in \mathbf{C}$ then $\overline{L} \in \mathbf{C}$. Show that $\mathbf{C}^{\mathbf{C}} = \mathbf{C}$. □
4. Suppose for a language $L$ there exists a polynomial time algorithm $A$ and $0 < \epsilon < 1$ such that for $x \in L$, $Pr(A(x) = 1) \geq 1 - \epsilon$ and for all $x \notin L$ $Pr(A(x) = 1) = 0$. What is the minimum value of $k$ for which $k$ fold repetition of the algorithm (call the new algorithm $A'$) will give $Pr(A'(x) = 1) \geq 1 - 1/2^n$ for $x \in L$ where $n = |x|$ and $Pr(A'(x) = 1) = 0$ for $x \notin L$ where $n = |x|$. (Observe that $A'$ runs in polynomial time.) □
5. *Reading exercise from Sipser:* Study the probability amplification lemma for the class **BPP**. That is, show that if for a language $L$ there exists a polynomial time algorithm $A$ and $0 < \epsilon < 1$ such that for $x \in L$, $Pr(A(x) = 1) \geq 1/2 + \epsilon$ and for all $x \notin L$ $Pr(A(x) = 0) \geq 1/2 + \epsilon$ then as in the above problem show that there exists a polynomially bounded $k$ (in the size of $x$) such that the algorithm $A'$ obtained by $k$ fold repetition of $A$ (with decision by majority in this case) satisfies if $x \in L$, $Pr(A'(x) = 1) \geq 1 - 1/2^n$ and if $x \notin L$, $Pr(A'(x) = 0) \geq 1 - 1/2^n$ where $n = |x|$. □
6. Show that **co-BPP**=**BPP** □
7. Show that if for some $i \geq 1$, $NC_{i+1} = NC_i$ then $NC = NC_i$. □
8. Show that **PolyL** does not have complete problems (w.r.t. $\leq_m^{log}$ reductions). Hence conclude that **P**$\neq$**polyL**. □
9. Show that 2SAT (satisfiability problem for boolean formula in conjuctive normal form with 2 literals per clause) is NL complete. □
10. Recall that in our definition of the class $NC_i$ we required that the gates have bounded fan-in (i.e., number of inputs: 2 input AND/OR gates were used). Suppose we allow unbounded fan-in, we can define $AC_i$ as the class of langauges accepted by polynomial sized $O(\log^i n)$ depth uniform circuits of unbounded fan-in. We define $AC = \bigcup_{i \geq 0} AC_i$. Show that $NC_i \subseteq AC_i \subseteq NC_{i+1}$. Hence conclude that $AC = NC$. □
11. Show that if **PSPACE**$\subseteq$ **P/Poly** then **PSPACE**$\subseteq \mathbf{\Sigma_2^p} \cap \mathbf{\Pi_2^p}$. □
12. Show that if **EXP**$\subseteq$ **P/Poly** then **EXP**$\subseteq \mathbf{\Sigma_2^p}$. □
13. Show that there exists undecidable languages in **P/Poly**. □
14. Show that if $\Sigma_i^p = \Sigma_{i+1}^p$ for some $i$ then **PH**$= \Sigma_i^p$ □
15. Show that if $3SAT \subseteq_m^p \overline{3SAT}$ then **PH=NP**. □
16. The problem of linear programming is of maximizing an objective function $\mathbf{c}^T\mathbf{x}$ subject to constraints $\mathbf{Ax} \leq \mathbf{b}$ where $\mathbf{c} \in \Re^n$, $\mathbf{x} =< x_1, ..x_n >$ variables, $\mathbf{A} \in \Re^{m \times n}$, $\mathbf{b} \in \Re^m$. Show that if the problem is in **NC** (in the sense solvabable with polynomial size circuits of polylogrithmic depth), show that **P**=**NC**. (Hint: Network flow). □
17. Show that the problem of determining whether the size of the smallest clique in a graph of $n$ vertices is $k$ is in $\Sigma_2^p$. Show that the problem of determining whether a given subset of $k$ vertices from a clique of smallest size in the graph is solvable in $\mathbf{P^{NP}}$. □