Assignment I

Jan. 2013 : Combinatorial Algorithms

1. Given a graph $G = (V, E)$ with weights w_v associated with each $v \in V$ and c_e to each edge $e \in E$. Consider the *Weighted vertex cover problem* of picking a collection $S \subseteq V$ such that $\sum_{v \in S} w_v$ is minimized subject to constraint that atleast one end point of each edge is picked in S. The *Weighted matching problem* is to find a collection T of edges maximizing $\sum_{e \in T} c_e$ subject to picking at most one edge adjacent to any vertex.
   - Find the LP Formulation of these problems and their duals.
   - Will the rounding algorithm discussed in class work for these problems? What about the primal dual algorithm? (Suppose you know that the degree of each vertex is at most k, will that help solving any of these problems?)
   - Analyze the situation when G is bipartite.
2. The set *Set Cover* problem is a generalization of vertex cover. Given a collection V of n elements and a collection $\mathbf{S} = \{S_1, S_2, ..., S_m\}$ of subsets of V, such that for each set $S_j \in \mathbf{S}$ has an associated weight w_j, we want to pick a collection of sets from $\mathbf{S}$ of minimum sum of weights such that every element in V appears in at least one of the sets picked. It may be assumed that each element in V appears in at least one and at most k sets in $\mathbf{S}$.
   - Write down the LP formulation for the problem and its dual.
   - How will you modify the rounding algorithm for vertex cover to suit this problem. What is the approximation ratio achieved in this case?
   - Modify the Primal-Dual algorithm for vertex cover to solve this problem. What will be the approximation ratio achieved?
3. The *edge cover (EC)* problem consists of selecting a minimum collection of edges from a connected undirected graph G such that for each vertex in G, at least one edge adjacent to it is selected. The *Maximum Independent Set (IS)* problem asks you to pick a maximum collection of vertices, no two of which are adjancent in G.
   - Are these problems duals of each other?
   - What can be deduced if the graph is bipartite? (Will the constraint matrix be TUM for any of these problems in the bipartite case?)
4. A thief enters a shop to find n items with price/kilogram $p_1, p_2..., p_n$ with and weight/kilogram $w_1, ...w_n$ respectively. The thief has a bag which can hold W kilograms. The question is which items (and how much of each) must be chosen to maximize the price of the theft without exceeding the bag capacity. Assume that no item gets exhausted by the theft. This problem is called the *Fractional Knapsack problem*
   - Formulate the problem as an LP. (Note that this is not an Integer LP)
   - Formulate the dual LP.
   - What is the optimal value of the objective function you can achieve for the dual LP? What are the corresponding values to the primal and dual variables for the optimum value? Note that looking at the dual, it is immediete what the optimal strategy must be for solving the original problem.
5. Given an $n \times n$ "chess-board", you must find the "most profitable" placing of rooks on the chess board such that no two rooks attack each other. The profit obtained on placing a rook in position (i, j) is $c_{i,j}$.
   - Formulate the problem as an Integer LP.
   - Formulate the dual LP.