Assignment II

Question 1: A relation R defined on a set X which is reflexive, antisymmetric (that is $(x, y) \in R$ and $(y, x) \in R$ if and only if $x = y$ for all $x, y \in X$) and transitive is called a **partial order**. For instance on the set of natural numbers $\mathbf{N} = \{0, 1, 2, 3...\}$ let R be the divisibility relation (that is, $(x, y) \in R$ if y is a multiple of x). A partial order R is called a **linear order** if for every $x, y \in X$ either $(x, y) \in R$ or $(y, x) \in R$. Note that $\mathbf{N}$ with divisibility is not a linear order.

Let $x, y \in X$. An element $a \in X$ is called the **meet** (or greatest lower bound) of x and y denoted by $a = x \wedge y$ if the following conditions hold: (i) $(a, x) \in R$, $(a, y) \in R$ (ii) If some other $a' \in X$ satisfies $(a', x) \in R$, $(a', y) \in R$ then $(a', a) \in R$.

Similarly An element $b \in X$ is called the **join** (or least upper bound) of x and y denoted by $b = x \vee y$ if the following conditions hold: (i) $(x, b) \in R$, $(y, b) \in R$ (ii) If some other $b' \in X$ satisfies $(x, b') \in R$, $(y, b') \in R$ then $(b, b') \in R$.

If $x \wedge y$ and $x \vee y$ exists for every pair $x, y \in X$, we say (X, R) is a **lattice**

1. if a and a' satisfy conditions (i) and (ii) in the definition of $a \wedge b$, show that $a = a'$. (Note that you need to use the fact is R is anti-symmetric here).

2. Let X be the set $\{1, 2, 3, 4, 5\}$ Let $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 4), (3, 5), (1, 4), (1, 5)\}$. Show that (X, R) is a partial order but not a lattice.

3. Suppose $Y = 2^X$, the set of all subsets of X. Let R be the inclusion relation in Y. That is, if A, B are subsets of X we say $(A, B) \in R$ when $A \subseteq B$. Show that R is a partial order. Is R a lattice? If so, what is $A \wedge B$ and $A \vee B$?

4. Show that $\mathbf{N}$ with R being the divisibility relation is a lattice. What is $x \vee y$ and $x \wedge y$ in this case?

5. Let $\mathbf{Q}$ and $\mathbf{R}$ represent the set of rationals and the set of real numbers respectively. Let R be the $\leq$ relation. Show that $(\mathbf{Q}, \leq)$ and $(\mathbf{R}, \leq)$ are linear orders. Show that they are also lattices. What is $x \wedge y$ and $x \vee y$ in these lattices?

6. Let (X, R) be a lattice. Let $S \subseteq X$. An element $a \in X$ is called the supremum of S (denoted by $sup(S)$ or $LUB(S)$) if a satisfies conditions (i) $(x, a) \in R$ for all $x \in S$ and (ii) whenever some other a' satisfies condition (i), $(a, a') \in R$. The infimum of S ($inf(S)$) is defined similarly. Show that for the set $S = \{x|x^2 < 2\}$, $sup(S)$ does not exist in $(\mathbf{Q}, \leq)$ whereas $sup(S)$ exists in $(\mathbf{R}, \leq)$. A lattice in which $sup(S)$ exists for every subset S of X is called a **complete lattice**

Question 2: Let (A, R) be a lattice. We will denote $\leq$ for R and write $a \leq b$ whenever $(a, b) \in R$ for any $a, b \in A$ be arbitrary.

1. Show that $a \leq b$ if and only if $a \wedge b = a$ and $a \vee b = b$.

2. Show that $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = b$.

3. Show that $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ and $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$ Give an example for a lattice where the inequalities are strict. A lattice is said to be **distributive** if equality holds for all a and b.

4. Let n be a natural number. By D_n we denote the set of divisors of n with the divisibility relation. For example $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ with $|$ denoting the divisibility relation. (that is, we write $a|b$ when b is a multiple of a). Draw the **Hesse diagrams** for the lattices D_{30}, D_{12}, D_{20} and D_{24}. Which among them are distributive? (Hint: There is a geometric way to figure out from the Hesse diagram whether a lattice is distributive. Learn it and then the problem becomes easy).

Question 3: Let $(L, \leq)$ be a lattice. Suppose there is an element $x_0 \in L$ such that $a \leq x_0$ for all $a \in L$, then we called x_0 the **greatest element** and denote it by 1. Similarly $y_0 \in L$ satisfies $y_0 \leq a$ for all $a \in L$, then y_0 is called the **least element** of L and is denoted by 0. $(L, \leq)$ is said to be **bounded lattice** if 0 and 1 exits in which case we denote L by $(L, \leq, 0, 1)$. A pair elements a and b in a bounded lattice L are said to be **complements** of each other if $a \wedge b = 0$ and $a \vee b = 1$.

1. Give an example for a bounded lattice L in which an element a has two complements b and b'.

2. Prove that in a distributive lattice if an element has complements b and b' then $b = b'$.

Question 4: Let f be a function from a bounded lattice $(L, \leq, 0, 1)$ to itself. We say f is **monotone** if $f(a) \leq f(b)$ whenever $a \leq b$.

1. Show that f is monotone if and only if $f(a \wedge b) \leq f(a) \wedge f(b)$ for all $a, b \in L$.

2. Consider the lattice $(\mathbf{R}, \leq)$. Give an example for a function that satisfies $x \leq f(x)$ for all $x \in \mathbf{R}$ but is not monotone.

Question 5: Let f be a monotone function on a complete lattice $(L, \leq, 0, 1)$. Consider the set $S = \{y : y \leq f(y)\}$. Let $x = sup(S)$. ($sup(S)$ must exist even if S is infinite because L is complete). Show that x satisfies $f(x) = x$. An element satisfying this equality is called a **fix point** of f. Hence this observation proves that every monotone function on a complete lattice must have a fix point. This result is a special case of **Tarski's fix point theorem**.

Question 6: This question develops an algebraic way of defining a lattice. Suppose L be a set with two binary operations $\wedge$ and $\vee$ defined on L satisfying: (i) $\wedge$ and $\vee$ are associative (ii) $\wedge$ and $\vee$ are commutative (iii) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a, b \in L$. Define the relation R on L as follows: $(a, b) \in R$ if and only if $a \wedge b = a$.

1. Show that if $a \wedge b = a$ if and only if $a \vee b = b$.

2. Show that R is a partial order. (verify reflexivity, anti-symmetry and transitivity).

3. Show that R is a lattice with $LUB(a, b) = a \vee b$ and $GLB(a, b) = a \wedge b$.

Question 7: Let $(L, \leq, 0, 1)$ be a distributive lattice. Suppose every $a \in L$ has a complement also, then L is called a **boolean lattice** (or a **boolean algebra**). Thus a boolean lattice is a complemented distributive lattice.

1. Which among the following lattices are boolean – $D_{30}, D_{12}, D_{105}, D_{25}$?

2. Suppose p_1, p_2, p_3 are prime numbers. Arugue that $D_{p_1 p_2 p_3}$ is a boolean lattice if and only if p_1, p_2 and p_3 are distinct prime numbers.

3. Let X be a finite set. Show that $(2^X, \subseteq, \emptyset, X)$ is a boolean lattice. (you may assume properties of set union and set intersection). Suppose $Y \subseteq X$, what is the complement of Y in this lattice?