Assignment I

1. Four bins B_1, B_2, B_3 and B_4 are placed on a table. The bin B_i has i RED and $i+1$ BLUE balls for $1 \leq i \leq 3$. A bin is drawn at random with the following probabilities: $p(B_1) = 1/2$, $p(B_2) = 1/4$, $p(B_3) = p(B_4) = 1/8$. Then a ball is drawn from the bin selected. Suppose the ball received is $BLUE$, what is the probability (conditioned on this fact) that the bin was B_1?

2. Use Jensen's inequality (and the concavity of log function) to prove Gibbs' inequality.

3. Consider a Binary Symmetric Channel (BSC) with error parameter ϵ called $BSC(\epsilon)$. The channel takes $X \in \{0,1\}$ as input and output $Y \in \{0,1\}$ with the following probabilities: $p(Y=0|X=0) = p(Y=1|X=1) = 1-\epsilon$. where $0 < \epsilon < 1/2$. Suppose $p(X=0) = p$, find: (a) $H(X|Y=0)$, (b) $H(Y|X=0)$, (c) $I(X,Y)$. (d) For what value of $p(X=0)$ is X, Y independent?

4. Let X, Y be random variable. Let $Z = f(Y)$ be a random variable that depends only on Y. Show that $I(X,Y) \geq I(X,Z)$. This shows that applying a deterministic function on Y (or data processing done on Y) cannot increase mutual information. When does equality hold?

5. Let X, Y be random variables and $Z = X \oplus Y$. Is there any relation between $I(X,Y)$ and $I(X,Z)$? Note that given X, all "information" about Z is revealed if Y is known and conversely. Is it true that $H(X|Y,Z) = 0$?

6. Suppose we want to have a ternary compression scheme (three symbols 0,1 and 2) instead of binary. What will be the revised expression for Kraft's inequality?

7. Suppose a source emits three symbols X_1, X_2, X_3 from $\Sigma = \{a, b\}$ such that $p(X_0 = a) = 1/4$, $p(X_0 = b) = 3/4$ and $p(X_{i+1} = a|X_i = a) = p(X_{i+1} = b|X_i = b) = 3/4$. What is the (joint) entropy of X_1, X_2, X_3? Show that $H(X_3|X_1, X_2) = H(X_3|X_2)$. What is the reason for this? Find the probability distribution for X_1 that would make the sequence stationary.