LOGIC for CS: Assignment I

1. Let $\mathcal{A}$ be an axiom set over variable set V. Let α, β, γ etc. be formulas over V. Prove the following:
   - $\mathcal{A} \models \alpha \vee \beta$ if and only if $\mathcal{A} \cup \{\neg\alpha\} \models \beta$.
   - $\mathcal{A} \models \alpha \rightarrow \beta$ if and only if $\mathcal{A} \cup \{\alpha \neg\beta\}$ is inconsistent.
   - $\mathcal{A}$ is categorical if and only if it is complete and consistent.
   - $\mathcal{A}$ is inconsistent if and only if it is complete and not categorical.

2. Let $\mathcal{A} = \{p_1 \rightarrow \neg p_2, (p_1 \wedge p_2) \rightarrow \neg p_3, (p_1 \wedge p_2 \wedge p_3) \rightarrow \neg p_4, ...\}$. As in the proof of the compactness theorem assume $\mathcal{A}_n$ is the restriction if $\mathcal{A}$ to formulas with only variables $p_1, p_2, ..p_n$. Consider the truth assignments $\tau_1 = TF, \tau_2 = FTF, \tau_3 = TFTF, \tau_4 = FTFTF, ...$
   - Does $\tau_n \models \mathcal{A}_n$ for each n?
   - If so, follow the steps of the compactness theorem to find a truth assignment that satisfies $\mathcal{A}$.
   - Is $\mathcal{A}$ categorical? if not, find another truth assignment that satisfies $\mathcal{A}$, different from the one you derived above.

3. Let $\mathcal{A}$ be an infinite collection of boolean formulas involving variables $p_1, p_2, .., p_n ...$ and let $\mathcal{A}_n$ be defined in the standard way. Can we conclude from the compactness theorem than if $\mathcal{A}$ is inconsistent, some $\mathcal{A}_n$ is inconsistent?

4. Translate the following argument into a set of formulas in propositional logic and determine whether the conclusion is valid: if 2 is prime, then it is the least prime. If 1 is prime, then 2 is not the least prime. 1 is not prime.
   - Is the conclusion "2 is the least prime" a valid consequence of the premises? (Note: One must not go out of propositional logic!).
   - Are the premises consistent? Complete? categorical?
   - Is it possible to find a formula (using the variables of the system) that is independent of the premises?

5. Apply Resolution to prove that $(p \vee \neg q \vee r) \wedge (q \vee r) \wedge (\neg p \vee r) \wedge (q \vee \neg r) \wedge (\neg q)$ is inconsistent.

6. Let A and B be finite sets with $|B| \geq |A|$. Let $R \subseteq A \times B$ be a relation. For each subset X of A, define $Neighbour(X) = \{y \in B : (x, y) \in R\}$. The Hall's theorem states that if for every finite subset X of A, the number of elements in $Neighbour(X)$ is at least as many as the number of elements in X, then there exists an injective map from A to B. Use the compactness theorem to prove that this theorem holds even if A and B are (countably) infinite sets.

7. (For students who have taken a course in topology earlier): In this question, we will develop a more general proof for the Compactness Theorem. Let $\mathcal{A}$ be a set of formulas over $\{p_0, p_1, p_2, ..\}$ Consider the space $\{T, F\}$ with the discrete topology. By Tichonoff theorem, $\{T, F\}^{\mathbf{N}}$ is Compact with respect to the product topology, where $\mathbf{N} = \{0, 1, 2, ..\}$. (Why?) Now, for any finite subset $\mathcal{F}$ of A, the set of truth assignments τ_F that satisfy $\mathcal{F}$ in $\{T, F\}^{\mathbf{N}}$ is a closed subset of the product topology (Prove!). Show that any truth assignment in the intersection of every τ_F for finite subsets $\mathcal{F}$ of $\mathcal{A}$ satisfies every formula in $\mathcal{A}$. Prove that the intersection is non-empty. (This proof explains the name compactness Theorem). Note that the proof generalizes to uncountable collection of formulas over uncountable collection of variables as well.