LOGIC for CS: Assignment II

1. * Note: Some errors had been reported on my earlier upload. This is the corrected version. If you find errors further, please report.

2. Write down $FOLG(=)$ axiom sets to characterize (up to isomorphism of canonical models) the following graphs/graph families:

    1. The complete (directed) graph of three vertices. Generalize this to any number k of vertices.
    2. All complete graphs with infinite number of vertices (you will need an infinite set of formulas to capture this specification. It is interesting to note that there is no way to capture all finite complete graphs - a fact which we will be able to easily prove a few lectures later).
    3. A cycle of length k.
    4. Graphs that do not contain any cycle of length k.
    5. Graphs that contain a cycle of length k for each $k \geq 2$.
    6. Graphs with in degree and out degree of each vertex having value one, but not containing any cycles.
    7. Set of all two colourable graphs (first read some very basic results in graph theory!).

3. A (commutative) ring (with unity) is a set $(A, +, *, 0, 1)$ with two binary operations $+$ and $*$ satisfying the following: $+$ is associative and commutative with identity 0, $*$ is commutative and associative with identity 1, $*$ distributes over $+$ and for each element a, there exists b such that $a + b = 0$. (That is, additive inverses exist). For Example $(\mathbf{Z}, +, *, 0, 1)$ is a ring. A ring with the additional property that all elements in A except the constant member 0 has multiplicative inverse (that is, inverse with respect to $*$) is called a field. For example the set of rational numbers is a field with respect to standard addition and multiplication. The set of real numbers too form a field.

    1. Write down first order axioms for rings and fields in $FO(\{=^2\}, \{+^2, *^2, 0^0, 1^0\})$ (including axioms for equality).
    2. Give examples for two non-isomorphic rings with 4 elements. Write down a first order formula that is true in one and false in the other.

4. Consider the axiomization of Abelian groups in $FO(\{=^2\}, \{+^2, *^2, 0^0\})$.
Let $M_1 = (A_1, \equiv_1, +_1, *_1, 0)$ and $M_2 = (A_2, \equiv_2, +_2, *_2, 0)$ be two models.

    1. Define the condition for a function $\pi : A_1 \longrightarrow A_2$ to be a group homomorphism.
    2. Prove the homomorphism theorem for Abelian groups. That is, if π is a homomorphism from M_1 to M_2 and if ϕ is a formula, then $M_1 \models \phi$ if and only if $M_2 \models \phi$.
    3. Let $M = (A, \equiv, +, *, 0)$ be a model for Abelian groups. Define the canonical model $\frac{M}{\equiv} = (\frac{A}{\equiv}, +, *, 0)$ and define $+$ and $*$ appropriately in the canonical model (strictly speaking, one should have written $\frac{+}{\equiv}$ and $\frac{*}{\equiv}$ etc., which is avoided here). Show that the functions $+$ and $*$ are well defined in the canonical model.
    4. Prove that for any formula $\phi \in \mathcal{F}(GROUPS)$, $M \models \phi$ if and only if $\frac{M}{\equiv} \models \phi$.

5. Consider the models for the logical system $FO(\{G^2, =^2\}\{0^0\}, \mathcal{A}_= \cup \mathcal{A})$. Here, we add a constant 0 to $FOLG(=)$. The additional axioms (other than equality axioms for the new symbol) are $\mathcal{A} = \{\forall x[(x \neq 0) \leftrightarrow \exists y G(y, x)], \forall x \forall y \forall z[G(x, y) \wedge G(x, z) \rightarrow (y = z)], \forall x \forall y \forall z[G(x, z) \wedge G(y, z) \rightarrow (x = y)], \forall x \exists y G(x, y)\}$. These axioms stipulate that each element has a unique successor and each element except 0 has a unique predecessor as well. Let $\mathbf{N} = \{0, 1, 2, ..\}$ and $\mathbf{Z}' = \{0', \pm 1', \pm 2'..\}$. Let M_1 be the model with $A_1 = \mathbf{N}$ and $G(x, y)$ interpreted $y = x + 1$. M_2 be the model with $A_2 = \mathbf{N} \cup \mathbf{Z}'$ such $R_2 = \{(i, i+1) : i \geq 0\} \cup \{(i', (i+1)') : i = 0, \pm 1, \pm 2, ...\}$

    1. Argue that the above logical system cannot have finite models.

2. Argue that M_1 and M_2 are models are non-isomorphic models for the logical system.
3. Find a model M_3 for the logical system over the set $\mathbf{N} \cup T$ such that $(i, i+1) \in R_3$ for each $i \geq 0$ and T is a finite set of k elements for some positive integer k.
4. Let $\phi_k = \exists y \forall x_1, ... \forall x_k [G(0, x_1) \wedge G(x_1, x_2) \wedge ... G(x_{k-1}, x_k) \rightarrow (y \neq x_k)]$ for $k \geq 1$. Does $M_1 \models \phi_k$ for each $k \geq 1$?

6. Consider the following $FOLG$ axioms (here, we use the symbol $<$ instead of G).
$\mathcal{A} = \{\forall x \neg(x < x), \forall x \forall y \forall z[(x < y) \wedge (y < z) \rightarrow (x < z)],$
$\forall x \forall y[(x < y) \vee (y < x)], \forall x \forall y \exists z[(x < y) \rightarrow ((x < z) \wedge (z < y))],$
$\forall x \exists y \exists z[(y < x) \wedge (x < z)]\}.$
    1. Argue that $\mathcal{A}$ does not have any finite models.
    2. Argue that any two countably infinite models of $\mathcal{A}$ are isomorphic.

7. Let $\mathbf{Z} = \{0, \pm 1, \pm 2, ..\}$. Let $\mathbf{Z}' = \{0', \pm 1', \pm 2'..\}$. Let $S = \{(i, i+1) : i = 0, \pm 1, \pm 2, ..\}$ and $T = \{(i', (i+1)') : i = 0, \pm 1, \pm 2, ..\}$. Let $M_1 = (\mathbf{Z} \cup \mathbf{Z}', S \cup T)$ and $M_2 = (\mathbf{Z}, S)$ be two $FOLG(=)$ structures.
    1. Is the map $\pi(i) = \pi(i') = i$ a homomorphism from M_1 to M_2?
    2. What is the minimum set of edges to be added M_1 to make π defined above a homomorphism?

8. Consider the $FOLG$ formula $\phi = (\forall x \exists y G(x, y) \wedge \exists x \forall y \neg G(x, y))$. Consider the logical system $L = FO(\{G^2\}, \{c^0, f^1\}, \{\})$. L is obtained by extending $FOLG$ by adding a constant c and a unary function f. Let $\phi' = \forall x \forall y[G(x, f(x)) \wedge \neg G(c, y)]$. Show that ϕ has an $FOLG$ model if and only if ϕ' has an L model.