LOGIC for CS: Assignment III

1. Consider the $FOLG$ axioms $\mathcal{A} = \{\forall x \exists y \exists z G(x,y) \wedge \neg G(x,z), \neg \exists x G(x,x)\}$. Let f and g be the single argument functions replacing y and z in the Skolem Normal form and let c be the constant added to the domain. (why z is not replaced by a two argument function?) Consider the model M with $A = \{a,b,c\}$, $R = \{(a,b),(b,c),(c,a)\}$, $f(a) = b, f(b) = c, f(c) = a$, $g(a) = c, g(b) = a, g(c) = b$
   1. Write down the equivalent formula in Skolem Normal form (in terms of f and g).
   2. Write down the Herbrand's universe $D(\mathcal{A})$.
   3. Define the binary relation G in $D(\mathcal{A})$ to yield a Herbrand model from M (following the proof of the completeness theorem).
   4. Suppose you think of the formula as an $FOLG(=)$ formula, how will you define equality relation $=$ in the Herbrand's universe to yield a model isomorphic to M, where equality in M is interpreted as the identity relation.
   5. Using the fact that $g(t) = f^2(t)$ in the above model, define G over $\{c, f(c), f^2(c), ...\}$ to yield a different (infinite) model.
2. Give a resolution proof for the inconsistency of the $FOLG$ axiom set $\mathcal{A} = \{\forall x \exists y \neg G(x,y), \exists x \forall y G(x,y)\}$.
3. This question describes how the proof of the completeness theorem can be extended to $FOLG(=)$. Let ϕ be a closed formula in $FOLG(=)$. Let the Skolem form of ϕ introduce a k argument function f (for some $k > 0$) and a constant c. Let $D(\phi)$ be the Herbrand universe for the formula. Let $M = (A, R, \equiv, f, c)$ be a model for the formula in Skolem form. We wish to define a homomorphism $\pi : D(\phi) \longrightarrow A$ and define G and $=$ over $D(\phi)$ in such a way that for each assignment τ for the variables into $D(\phi)$, $(A, R, \equiv, f, c, \pi\tau) \models \phi'$ only if $(D(\phi), G, =, f, c, \tau) \models \phi'$ for each sub formula ϕ' of ϕ.
   1. Define the map π, and define G and $=$ in $D(\phi)$.
   2. Prove that with your definitions, the $(A, R, \equiv, f, c, \pi\tau) \models \phi'$ only if $(D(\phi), G, =, f, c, \tau) \models \phi'$ holds for each sub formula ϕ' of ϕ and for each assignment τ for the variables into $D(\phi)$.
4. Consider the formula $\phi = \forall x \exists y ((x \neq y) \wedge G(x,y))$ in $FOLG(=)$, Consider the model M with $A = \{a,b,c\}$, $R = \{(a,b),(b,c),(c,a)\}$ and $f(a) = b, f(b) = c, f(c) = a$. Define G and $=$ in the Herbrand universe as in the proof of the completeness theorem (outlined in the previous question) for $FOLG(=)$.
5. Recall that two (not necessarily isomorphic) models M_1 and M_2 for any first order theory are *first order equivalent* if for each first order formula ϕ, $M_1 \models \phi$ if and only if $M_2 \models \phi$. Consider the model $(\mathbf{N}, <)$ where $\mathbf{N} = \{0, 1, 2, ..\}$ and $<$ the normal less than relation over $\mathbf{N}$. Show that there is a model M with an infinite descending chain that is first order equivalent to $\mathbf{N}$. (An infinite descending chain is an infinite sequence $a_1, a_2, ...$ of (distinct) elements such that $a_{i+1} < a_i$ for each integer $i > 0$.).
6. Consider the $FOLG(=)$ model consisting of the set of integers $\mathbf{Z}$ with $G(x,y) = 1$ if and only if $y = x + 1$ or $y = x - 1$. Show that there is a disconnected model that is (first order) equivalent to this model. (Hint: Introduce constants c and d to $FOLG(=)$ and write down the conditions c and d are at distance at least k for each positive integer k).
7. Let $\mathcal{A}$ be a set of axioms over $FOLG(=)$ such that there is an algorithm B that systematically generates each finite subset of $\mathcal{A}$, one after another in some order, running in an infinite loop. (Note that $\mathcal{A}$ need not be finite.) Moreover, assume that $\mathcal{A}$ is categorical.
   1. Show that for each $FOLG(=)$ formula ϕ, there is a resolution proof that either ϕ is a logical consequence of $\mathcal{A}$ or ϕ can never be true in any model satisfying $\mathcal{A}$.
   2. Show that you can design an algorithm (using B as a subroutine) to decide whether a given formula ϕ is a logical consequence of $\mathcal{A}$ or not.