LOGIC for CS: Assignment IV

1. A real number is called an *algebraric number* if the number is the root of some polynomial with rational coefficients. Show that the set of algebraric numbers is countabily infinite. (Hint: A polynomial can have only finitely many roots).

2. Show that there is a bijection between the sets $[0, 1)$ and $[0, 1]$ of real numbers (Hint: Use indirect arguments).

3. Show that the power set of natural numbers and the power set of integers have the same cardinality.

4. Show that the power set of real numbers have a cardinality greater than the set of real numbers.

5. Define a monotone function from the set $[1, 2] \cap \mathbf{Q}$ to itself that does not have a fix point. [Hint: If $x < \sqrt{2}$ set $f(x) = x + \frac{1}{2^{k_x}}$ where k_x is the smallest positive integer t such that $x + \frac{1}{2^t} < \sqrt{2}$ etc....]

6. Let $(L, \leq)$ be a complete lattice. Let 0 be the symbol indicating the smallest element in L. (Why should such an element exist?) Let E be the interesection of all closed sets containg 0. (That is, in the definition of closed set in the Bourbaki Witt theorem set $x_0 = 0$). Show that $sup(E)$ is the least fix point of f. (That is, if $sup(E) = t$, then t is a fix point of f and if t' is any other fix point of f, then $t \leq t'$).

7. Consider the set of all relations on a set A. Given a relation R defined on A, define $f(R) = R \cup \{(a, c) : \exists b \in A, (a, b), (b, c) \in R\}$. Thus f s a map from $2^{A \times A}$ to $2^{A \times A}$. (f is a function that takes as input a relation and gives as output a relation.)

    1. Show that $A \times A$ with set inclusion is a complete lattice and f is a monotone function on this lattice.
    2. Show that fix points of f are precisely transitive relations over A.
    3. The smallest closed set containing R (see previous question for terminology) is called the transitive closure of R.

8. A relation R' on a set A is an extension of a relation R on A is $R \subseteq R'$. Let R be any partial order on A, show that there exists a linear order R' that extents R. (Consider thee set of all partial orders that extents R. Show that such a maximal partial order must exist using Zorn's lemma. Show that such a maximal extension must be a linear order).

9. Show that every chain in a paritially ordered set is contained in a maximal chain (Hint: Zorn's Lemma).