1. Can there exist a function f on a *complete lattice* $(L, \leq)$ such that for all $x \in L$, $x < f(x)$? [3]

   Soln: No. A complete lattice must have a maximum element 1. But $f(1) > 1$ is impossible.

2. In **every** *poset* $(P, \leq)$ without maximal elements, does there exist a function f such that $x < f(x)$ for all $x \in P$? [3]

   Soln: Yes, because for each $x \in P$, $GT(x) = \{y \in P : x < y\}$ is non-empty for otherwise x would be maximal. Using axiom of choice, for each x, we can choose an $f(x) \in GT(x)$.

3. Let f be a *progressive* function on a *complete lattice* $(L, \leq)$. Can there exist a non-empty chain $C \subseteq L$ such that whenever $x \in C$, $f(x) \in C$, but $\sup(C) \notin C$? [3]

   Soln: Yes. Consider the complete lattice $\overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ with the normal $\leq$ relation and the progressive function $f(x) = x + 1$. The chain $C = \{0, 1, 2, 3, ...\}$ satisfies $f(x) \in C$ whenever $x \in C$, but $sup(C) = +\infty \notin C$.

4. Let f be an injective map from a set A to another set B. Let g be an injective map from B to A. Let C be a subset of A such that $A - C = g(B - f(C))$. How will you define a bijection h between A and B? [3]

   Soln: Define $h(x) = f(x)$ if $x \in C$ and $h(x) = g^{-1}(x)$ if $x \in A - C$.

5. *Without* using the Bourbaki Witt Theorem, prove that on a *complete lattice* $(L, \leq)$, a progressive function f has a fix point. [3]

   Soln: Let x be the maximum element in L. $f(x) \geq x \Rightarrow f(x) = x$

6. A *poset* $(W, \leq)$ is well ordered if for each non-empty subset S of W, $\inf(S)$ exists and $\inf(S) \in S$. Is it true that every well ordered poset is a lattice? [3]

   Soln: Yes. Let $x, y \in W, x \neq y$. Since the set $\{x, y\}$ has a minimum element, one of the elements, say x must be smaller than the other one; that is $x < y$. But then $sup(\{x, y\}) = y$ and $\inf(\{x, y\}) = x$ and W satisfies the lattice requirements.

7. Consider the set of all binary sequences $A = \{(a_0, a_1, a_2, ...) : a_i \in \{0, 1\}\}$. Show that A is uncountably infinite. [3]

   Soln: Straightforward diagonal argument. Assume that $a^0, a^1, a^2, ..$ be an enumeration of all sequences, where each $a^i = (a^i_1, a^i_2, a^i_3..)$ is an infinite binary sequence. Construct the diagonal sequence $b = (b_0, b_1, b_2, ..)$ where $b_i = 1 - a^i_i$. It is easy to see that b differs from a^i in the value of the i^{th} term, and hence not part of the enumeration, contradicting the assumption that all binary sequences can be enumerated.

8. Let $(P, \leq)$ be a poset. A subset S of P is an *antichain* if for each $x, y \in S$, **neither** $x < y$ **nor** $y < x$. Does *every* poset contain a *maximal* antichain? [3]

   Soln: Yes. Apply Zorn's Lemma. Consider the set $A(P)$ of all antichains in P with the subset relation $\subseteq$. If $\{C_i\}_{i \in I}$ is a collection of antichains, such that for each $i, j \in I$, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$, then it is easy to see that $\cup_{i \in I} C_i$ is an antichain as well (why? - this statement could be succintly stated as: "union of a chain of anti-chains is an anti-chain!"). Thus $(A(P), \subseteq)$ is a chain complete poset. In particular, every chain (of antichains) has an upper bound (their union). It follows by Zorn's lemma that $A(P)$ contains a maximal element.

9. Let f be a progressive function on a chain complete poset $(P, \leq)$. Let $x_0 \in P$ A subset A of P is said to be open if 1) $x_0 \in A$, 2) Whenever $x \in A$, $f(x) \in A$ and 3) for any chain $C \subseteq A$, $\sup(C) \in A$. Let E be the intersection of all open subsets of P. Can we conclude that for each $x \in E$, $x_0 \leq x$? [3]

   Soln: Yes. Let $Q = \{x \in E : x \geq x_0\}$. If we prove that Q is open, it follows that $Q = E$ and proves the claim (why?). 1) $x_0 \in Q$ by definition of E and Q. 2. If $x \in Q$, we have $x \geq x_0$ and since

f is progressive, we have $f(x) \geq x$. Thus we have $f(x) \geq x \geq x_0$ ensuring that $f(x) \in Q$. Finally, 3. if C is a chain in Q and let $x = sup(C)$. For each $c \in C$, $c \geq x_0$. Hence $sup(C) \geq x_0$ and thus $c \in Q$. Thus Q is open, proved.

10. **Either** construct a Herbrand Model **or** show a resolution proof for the unsatisfiability of the $FOLG$ formula $\exists x \forall y \exists z (G(x,y) \wedge \neg G(x,z))$. [4]

Soln: The functional form is $\phi(y) = \forall y (G(c,y) \wedge \neg G(c, f(y)))$, Herbrand Universe $\mathcal{D}(\phi) = \{c, f(c), f^2(c), ..\}$ and Herbrand expansion
$\mathcal{H}(\phi) = \{\phi(c), \phi(f(c)), \phi(f^2(c)), ...\} = \{G(c,c) \wedge \neg G(c, f(c)), G(c, f(c)) \wedge \neg G(c, f^2(c)), ...\}$.
It is now an easy resolution to prove unsatisfiability of $\mathcal{H}(\phi)$

11. Consider the following $FOLG(=,0)$ axioms to capture $T = \{.. -3, -2, -1, 0\}$ with G modeling a successor function defined by $succ(x) = x + 1$: 1) 0 has a unique predecessor, but no successor. 2) Every non-zero element has a unique successor and predecessor different from itself. [3x3]

   1. Formulate the above properties in $FOLG(=)$.
      Soln: 1. $\exists x \forall y \forall u (G(x,0) \wedge (G(y,0) \Rightarrow (y = x)) \wedge \neg G(0,u))$
      2. $\forall x [(x \neq 0) \Rightarrow \exists y \exists z \forall p \forall q \{G(x,y) \wedge G(z,x) \wedge \ (G(x,p) \Rightarrow (p = y)) \wedge (G(q,x) \Rightarrow (q = z))\}]$
   2. Give a model satisfying the above axioms that is not isomorphic to T.
      Soln: (A, R) with $A = T \cup \{a, b\}$, $R = \{(i-1, i) : i < 0, i \text{ integer }\} \cup \{(a,b), (b,a)\}$.
   3. Show that it is impossible to categorically axiomize T by adding more $FOLG(=)$ axioms to the above axiom set.
      Soln: Suppose $\mathcal{A}$ is a collection of $FOLG(=)$ axioms that categorically axiomize the model T. Consider the extension of $FOLG(=)$ with constants 0 and c yielding $FOLG(=,0,c)$. Add axioms $\phi_0 = \forall x \neg G(0,x)$, $\phi_1 = \forall x_1 G(x_1, 0) \Rightarrow (x_1 \neq c)$, $\phi_2 = \forall x_1 \forall x_2 G(x_2, x_1) \wedge G(x_1, 0) \Rightarrow (x_2 \neq c)$, ... Basically the axioms stipulate that 0 has no successor and c has no path to zero in finitely many steps. Since any finite subset of these added set is satisfied by our standard model T, by compactness theorem it follows that $\mathcal{A} \cup \{\phi_0, \phi_1, ...\}$ must have a model. However, T does not satisfy $\mathcal{A} \cup \{\phi_0, \phi_1, ...\}$ (why) and hence there must be some other model, not isomorphic to T, which is satisfied by $\mathcal{A} \cup \{\phi_0, \phi_1, ...\}$. But this model satisfies $\mathcal{A}$ as well. Hence, $\mathcal{A}$ fails to categorically axiomize T, proving the claim.