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# Lecture 1: Propositional Calculus

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In this lecture, we will study the logic of propositions. A proposition is a statement which takes value *true* or *false*. We will use propositional variables like $p, q, r$ to denote propositions. Propositional formulas are constructed from variables using the *logical connectives* $\wedge, \vee, \rightarrow$ and $\neg$. Once the *truth values* of the variables of a formula are known, the truth value of the formula can be evaluated. These notions are formalized below.

## Syntax of Propositional Calculus

Let $V$ be a collection of propositional variables. The set of Boolean (or propositional) formulas over $V$ denoted by $\mathcal{F}_V$ are inductively defined as follows:

- if $\phi \in V$ then $\phi \in \mathcal{F}_V$.
- if $\phi, \psi \in V$, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$, $(\neg\phi)$, $(\neg\psi)$ are in $\mathcal{F}_V$.

**Example 1.** *If $V = \{p, q, r\}$ the $(p \wedge (q \rightarrow r))$, $(\neg q \rightarrow (p \vee q))$ etc. are formulas.*

The normal convention is that $\neg$ has the highest precedence among the connectors $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ and $\neg$. $\wedge$ has higher precedence over $\vee$, which in turn has higher precedence over $\rightarrow$ and $\leftrightarrow$. $\wedge$ and $\vee$ are left associative, whereas, $\neg$, $\leftrightarrow$ and $\rightarrow$ are right associative. This allows parenthesis to be omitted. For instance $p \vee q \rightarrow \neg r \wedge p$ denotes $(p \vee q) \rightarrow ((\neg r) \wedge p)$.

Formulas must be given "life" by assigning truth values. This is our next objective. We will use 1 and 0 instead of *true* and *false*.

## Semantics of Propositional Calculus

Given a variable set $V$. A *Truth assignment* for $V$ is a map $\tau : V \longrightarrow \{0, 1\}$ We can extend $\tau$ inductively into a function from $\mathcal{F}_V$ to $\{0, 1\}$ (with a little abuse of notation) as follows:

- $\tau(\phi)$ is already defined if $\phi \in V$.
- $\tau(\phi \wedge \psi) = 1$ if both $\tau(\phi) = 1$ and $\tau(\psi) = 1$, 0 otherwise.
- $\tau(\phi \vee \psi) = 1$ if either $\tau(\phi) = 1$ or $\tau(\psi) = 1$, 0 otherwise
- $\tau(\phi \rightarrow \psi) = 0$ if $\tau(\phi) = 1$ or $\tau(\psi) = 0$, 1 otherwise
- $\tau(\phi \leftrightarrow \psi) = 1$ if $\tau(\phi) = \tau(\psi)$, 0 otherwise
- $\tau(\neg\phi) = 1$ if $\tau(\phi) = 0$, 1 otherwise.

**Example 2.** *Let $V = \{p, q, r\}$. Let $\tau(p) = \tau(q) = 1$ and $\tau(r) = 0$. Then $\tau(q \rightarrow r) = 0$, $\tau(p \wedge (q \rightarrow r)) = 0$ $\tau(p \leftrightarrow q) = 1$ etc.*

**Definition 1.** *A formula is said to be satisfiable if there is a truth assignment to its variables that makes the formula evaluate to true. A set of formulas is satisfiable if there is a truth assignment that satisfies every formula in the set. These notions and some other related ones are formalized below.*

- *For $\phi \in \mathcal{F}_V$, We say $\tau \in \{0,1\}^V$ satisfies $\phi$ if $\tau(\phi) = 1$. Define $\mathcal{M}_\phi = \{\tau \in \{0,1\}^V : \tau(\phi) = 1\}$. This is the collection of all truth assignments to $V$ that satisfies $\phi$.*
- *$\phi \in \mathcal{F}_V$ is said to be satisfiable if $\mathcal{M}_\phi \neq \emptyset$. A formula is satisfiable if there is at least one truth assignment that satisfies it.*
- *For $\mathcal{A} \subseteq \mathcal{F}_V$, $\mathcal{M}(\mathcal{A}) = \bigcap_{\phi \in \mathcal{A}} \mathcal{M}_\phi$. This the collection of truth assignments that satisfies all formulas in $\mathcal{A}$. Each $\tau \in \mathcal{M}(\mathcal{A})$ is called a* **model** *for $\mathcal{A}$.*
- *$\mathcal{A} \subseteq \mathcal{F}_V$ is said to be satisfiable or* **consistent** *if $\mathcal{M}(\mathcal{A}) \neq \emptyset$. Thus a set is consistent iff there is at least one truth assignment that satisfies every formula in the set. $\mathcal{A}$ is said to be inconsistent if it is not consistent.*
- *$\mathcal{A} \subseteq \mathcal{F}_V$ is said to be* **categorical** *(or sometimes* **complete***) if $|\mathcal{M}(\mathcal{A})| \leq 1$. That is, either $\mathcal{A}$ is inconsistent or there is a unique $\tau : V \longrightarrow \{0,1\}$ that satisfies $\mathcal{A}$.*
- *$\phi \in \mathcal{F}_V$ is said to be* **independent** *of $\mathcal{A} \subseteq \mathcal{F}$ if both $\mathcal{A} \cup \{\phi\}$ and $\mathcal{A} \cup \{\neg\phi\}$ are consistent. That is, there exists truth assignments $\tau_1, \tau_2 \in \mathcal{M}(\mathcal{A})$ such that $\tau_1(\phi) = \tau_2(\neg\phi) = 1$ and $\tau_1(\neg\phi) = \tau_2(\phi) = 0$.*
- *$\psi \in \mathcal{F}_V$ is said to be a* **logical consequence** *of $\phi$ if every $\tau \in \mathcal{M}_\phi$ satisfies $\psi$. That is, whenever a truth assignment makes $\phi$ true, it should make $\psi$ also true. In this case we write $\phi \Rightarrow \psi$.*
- *$\psi \in \mathcal{F}_V$ is said to be* **logically equivalent** *to $\phi \in \mathcal{F}_V$ if for every $\tau \in \{0,1\}^V$, $\tau(\phi) = \tau(\psi)$. That is, every truth assignment to the variables give the same truth value to both $\psi$ and $\phi$. In this case, we write $\phi \Leftrightarrow \psi$.*
- *$\phi \in \mathcal{F}_V$ is said to be a logical consequence of $\mathcal{A} \subseteq \mathcal{F}_V$ if every $\tau \in \mathcal{M}(\mathcal{A})$ satisfies $\psi$. In this case we write $\mathcal{A} \models \psi$.*
- *$\mathcal{A}, \mathcal{A}' \subseteq \mathcal{F}_V$ are said to be logically equivalent if $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{A}')$. That is, the set of truth assignments (models) that satisfy all formulas in $\mathcal{A}$ and $\mathcal{A}'$ are exactly the same.*
- *$\phi \in \mathcal{F}_V$ is a* **tautology** *if $\mathcal{M}_\phi = \{0,1\}^V$. That is, $\tau(\phi) = 1$ all truth assignments $\tau \in \{0,1\}^V$.*
- *$\phi$ is* **contradictory** *if $\mathcal{M}_\phi = \emptyset$. That is $\phi$ is always false. Note that $\phi$ is a tautology if and only if $\neg\phi$ is contradictory.*

Note that the sets $\mathcal{A}$ and $\mathcal{A}'$ in these definitions could contain infinitely many formulas from $\mathcal{F}$.

**Example 3.** *Let $V = \{p, q, r\}$ and $\mathcal{A} = \{p \to q, q \to r, \neg r \vee \neg q\, q \vee r\}$. The set $\mathcal{A}$ is consistent as $\tau(p) = \tau(q) = 0, \tau(r) = 1$ satisfies $\mathcal{A}$. The set is categorical as no other truth assignment satisfies the set. $\neg p$, is an example of a logical consequence of $\mathcal{A}$. Since the set is categorical, there is no formula in $\mathcal{F}_V$ that is independent of $\mathcal{A}$ (why?).*

**Example 4.** *The set* $\mathcal{A} = \{p_1 \vee p_2, p_2 \vee p_3, p_3 \vee p_4, ...\}$ *over* $V = \{p_1, p_2, ...\}$ *is consistent.* $\tau(p_i) = 1$ *for all* $i$ *satisfies* $\mathcal{A}$. $\mathcal{A}$ *is not categorical (why?).* $\neg p_2 \rightarrow (p_1 \vee p_3)$ *is a logical consequence of* $\mathcal{A}$ *(why?). For each* $i$*, the formula* $p_i \in \mathcal{F}_V$ *is independent of* $\mathcal{A}$ *(why?).*

**Exercise 1.** *Show that if* $\mathcal{A} \subseteq \mathcal{F}_V$ *is categorical, then for every* $\phi \in \mathcal{F}_V$ *either* $\mathcal{A} \cup \{\phi\}$ *is inconsistent or* $\mathcal{A} \cup \{\neg\phi\}$ *is inconsistent. Hence there is no* $\phi \in \mathcal{F}_V$ *that is independent of a complete set* $\mathcal{A} \subseteq \mathcal{F}_V$

**Exercise 2.** *Let* $V = \{p, q, r\}$ *Give an example for an consistent and complete set* $\mathcal{A} \subseteq \mathcal{F}_V$ *and formula* $\phi \in \mathcal{F}_V$ *such that both* $\mathcal{A} \cup \{\phi\}$ *and* $\mathcal{A} \cup \{\neg\phi\}$ *are inconsistent.*

**Definition 2.** *Let* $\phi, \psi \in \mathcal{F}_V$.

- $\phi$ **tautologically implies** $\psi$ *if the formula* $\phi \rightarrow \psi$ *is a tautology.*
- $\phi$ *is* **tautologically equivalent** *to* $\psi$ *if* $\phi \leftrightarrow \psi$ *is a tautology.*

**Exercise 3.** *Show that a* $\psi$ *is a logical consequence of* $\phi$ *if and only if* $\phi$ *tautologically implies* $\psi$*. Hence the notation for logical consequence* $\phi \Rightarrow \psi$ *will be used whenever* $\phi \rightarrow \psi$ *is a tautology.*

**Exercise 4.** *Show that a* $\psi$ *is logically equivalent to* $\phi$ *if and only if* $\phi \leftrightarrow \psi$ *is a tautology (that is* $\phi$ *is tautologically equivalent to* $\psi$*). Hence we write* $\phi \Leftrightarrow \psi$ *whenever when* $\phi \leftrightarrow \psi$ *is a tautology.*

The notions of tautological implication and logical consequence mean exactly the same concept in view of the exercises above. This notion has central importance in *deductions* which we will see as we proceed further through these notes.

It might appear strange as to why two different definitions were given for the same idea. The reasons are somewhat subtle.

In modelling a real system, we start typically with a set $\mathcal{A}$ of formulas which we know are true for the system (called *axioms* or *postulates* for the system) and try to find out what other formulas are true in the system - i.e., find out the formulas which are logical consequences of the axioms. However, this requires working with all possible truth assignments to the variables - an impossible task when there are infinitely many variables. Even with finitely many variables, trying out all possible truth assignments is a brute force approach and not practically useful.

Instead a practical way to solve the problem is to do the following. Suppose we want to prove that $\mathcal{A} \models \phi$. We try to identify a formula $\psi$ for which we already know that $\mathcal{A} \models \psi$ (for example $\psi$ could be an axiom in $\mathcal{A}$). Now, if we can show that $\psi \Rightarrow \phi$, then it will follow (will be shown soon) that $\phi \in \mathcal{M}(\mathcal{A})$. This is essentially the fundamental notion of *deduction* commonly found in mathematical reasoning and deduction algorithms. Note that the deduction step involves only two formulas $\phi$ and $\psi$. They would have only a few variables and hence would be simpler to handle. All known theorem proving techniques essentially use the technique of deduction in one form or the other to discover new truths about the system from what is already proved.

To execute this plan, one needs a good database of "standard" tautological implications. Some of them are developed in the following exercises. The following section formalizes the notions of axiomatic systems, deduction, theoremhood etc.

**Example 5.** *The formulas* $p \vee \neg p$, $p \wedge (p \rightarrow q) \Rightarrow q$ *(called* **Modus Ponens***),* $(p \rightarrow q) \wedge \neg q \Rightarrow \neg p$ *(called* **Modus Tollens***),* $(p \vee q) \wedge \neg p \Rightarrow q$ *(***disjunctive syllogism***)* $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$ *(***hypothetical syllogism***),* $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$ *(***Law of counter positive***),* $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$, $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ *(De-Morgan's Laws),* $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$, $\neg\neg p \Leftrightarrow p$ *(Law of double negation) etc. are standard examples of tautologies.* $\wedge$, $\vee$ *are* **associative**, **commutative** *and* **distribute** *over each other. The following properties:* $(p \wedge p) \Leftrightarrow (p \vee p) \Leftrightarrow p$, $(p \wedge q) \Rightarrow p$, $(p \wedge q) \Rightarrow q$ *and may be collectively be called the* **absorption properties**.

*As an example, to verify modus ponens, suppose* $\tau(p \wedge (p \rightarrow q) \rightarrow q) = 0$ *Then* $\tau(q) = 0$ *and* $\tau(p \wedge (p \rightarrow q)) = 1$. *The latter requires* $\tau(p) = 1$ *and* $\tau((p \rightarrow q)) = 1$. *But as* $\tau(q) = 0$, *for* $\tau((p \rightarrow q)) = 1$, *we need* $\tau(p) = 0$ *which is a contradiction. (Another standard verification method is using truth tables) The other formulas may be verified similarly.*

Another important technique is **substitution** Suppose you replace a variable with any formula uniformly in a tautology $\phi$, then the resultant formula is also a tautology. For instance in the tautology $p \vee \neg p$, if we substitute $p$ everywhere with $(p \rightarrow (q \vee r)$, we get $(p \rightarrow (q \vee r) \vee \neg(p \rightarrow (q \vee r))$ which also is a tautology. This is because $\phi$ evaluates to 1 under any values to the variables. Similarly, since equivalent formulas evaluate to the same truth value, any formula in an expression can be substituted by a logically equivalent formula without affecting truth values. These rules are called **substitution laws** and are summarized below for easy reference.

**Theorem 1** (Law of Substitution)**.**

1. *If* $\phi \in \mathcal{F}_V$ *is a tautology. Let* $p \in V$ *be a variable in* $\phi$. *Let* $\psi \in \mathcal{F}_V$. *Then formula obtained by substitution of all occurrence of* $p$ *with* $\psi$ *in* $\phi$ *also is a tautology.*
2. *Let* $\phi, \alpha.\beta \in \mathcal{F}_V$, *if* $\alpha$ *is a subformulas of* $\phi$ *and* $\alpha \Leftrightarrow \beta$, *Let* $\phi'$ *be the formula obtained by replacing of* $\alpha$ *with* $\beta$ *in* $\phi$. *Then for every* $\tau \in \{0,1\}^V$, $\tau(\phi) = \tau(\phi')$.

## Classical Deduction

**Definition 3.** *Let* $\mathcal{A} \subseteq \mathcal{F}_V$. *Define the set* $\mathcal{V}(\mathcal{A}) = \{\phi \in \mathcal{F}_V : \mathcal{A} \models \phi\}$. *This set is the collection of all formulas which are logical consequences of* $\mathcal{A}$.

**Definition 4.** *Let* $\tau \in \{0,1\}^V$. *Define* $\mathcal{F}(\tau) = \{\phi \in \mathcal{F}_V : \tau \models \phi\}$. $\mathcal{F}(\tau)$ *is the collection of all formulas which are satisfied by* $\tau$. *Let* $\mathcal{T} \subseteq \{0,1\}^V$. *Define* $\mathcal{F}(\mathcal{T}) = \bigcap_{\tau \in T} \mathcal{F}(\tau)$ *to be the collection of all formulas satisfied by every truth assignment in* $T$.

Axiomatic deduction methods were known right from the time of the ancient Greeks and an treatment of geometry can be found in Euclid's book "The Elements". The assumptions about the system under study were postulated as axioms for the system and logical consequences of these axioms were derived using a set of "standard" deduction rules. The derived consequences are called theorems. The notion of algorithmic (automated) deduction did not exist during those times and deductions had to be done manually. In this section, we shall discuss a few deduction methods well known from from the ancient times. $\phi, \psi \in \mathcal{F}_V$

**Theorem 2** (Laws of Deduction)**.** *Let* $\mathcal{A} \subseteq \mathcal{F}_V$ *and* $\phi, \psi \in \mathcal{F}_V$. *Then:*

- *(Deduction Theorem): $\mathcal{A} \models \phi$ and $(\phi \Rightarrow \psi)$ then $\mathcal{A} \models \psi$.*
- *(Law of Implication): $\mathcal{A} \models (\phi \Rightarrow \psi)$ if and only if $\mathcal{A} \cup \{\phi\} \models \psi$*
- *(Method of Contradiction): $\mathcal{A} \cup \{\neg\phi\}$ is inconsistent, then $\mathcal{A} \models \phi$.*

*Proof.* The first statement is proved here and the rest are left as exercises. Suppose $\mathcal{A} \models \phi$. Suppose $\tau \in \{0,1\}^V$ satisfies $\tau \models \mathcal{A}$. Then, by hypothesis, $\tau \models \phi$. As $\phi \Rightarrow \psi$ is a tautology, $\tau(\psi) = 1$ (why?). Hence $\tau \models \psi$. This proves the first part. The other parts are proved similarly. $\square$

**Example 6.** *Let $\mathcal{A} = \{p \to q, q \to \neg(r \to p), \neg r \vee \neg q\}$. Here is a deduction for $\neg p$:*

1. *$q \to \neg(r \to p)$ (axiom)*
2. *$q \to \neg(\neg r \vee p)$ (Substitution: $(r \to p) \Leftrightarrow (\neg r \vee p)$).*
3. *$q \to (r \wedge \neg p)$ (Substitution: De-Morgan's equivalence).*
4. *$p \to q$ (axiom)*
5. *$p \to (r \wedge \neg p)$ (Hypothetical Syllogism from 4,3)*
6. *$(\neg p \vee (r \wedge \neg p)$ (Substitution)*
7. *$(\neg p \vee r) \wedge (\neg p \vee \neg p)$ (Distributive law).*
8. *$\neg p \vee \neg p$ (Absorption law)*
9. *$\neg p$. (Absorption law)*

**Exercise 5.** *It is true that Mike has a bike. If Mike has a bike, then Mike can't have a car. If mike does not have a car, then Mike can't travel long distance. Either Mike travels long distance or Mike is a sportsman. Formulate the above statements in propositional logic. Is the statement "Mike is a sportsman" a valid consequence of these statements? If so, find a deduction for the statement based on the above laws. Are these statements consistent? Do they form a categorical set?*

**Exercise 6.** *Either cat fur or dog fur was found at the scene of the crime. If dog fur was found at the scene of the crime, officer Thompson had an allergy attack. If cat fur was found at the scene of the crime, then Macavity is responsible for the crime. But officer Thompson didnt have an allergy attack, and so therefore Macavity must be responsible for the crime. Is the conclusion correct?*