# Assignment I

1. Give a resolution proof for the unsatisfiability of the set of propositaional formulas $\{(p_1 \vee \neg p_2 \vee p_3), (p_2 \vee p_3), (\neg p_1 \vee p_3), (p_2 \vee \neg p_3), (\neg p_2)\}$. □

2. (For non CS Students): Understand the notions of an algorithm, polynomial time algorithm, NP-hard problem and NP-complete problem. Understand the statement of the famous "Cook's Theorem" and the notion of how a problem can be shown to be NP-complete by showing that there exits a polynomial time reduction from a problem already known to be NP-hard. □

3. Suppose each clause in a finite set of propositional clauses has atmost two literals, show that resolution gives a polynomial time algorithm for checking satisfiability of the clause set. (Hint: Each clause can be written in the from $(p_i \rightarrow p_j)$ (or $(p_i \rightarrow \neg p_j)$ or $(\neg p_i \rightarrow p_j)$ or $(\neg p_i \rightarrow \neg p_j)$). This may be thought of as a dependency relation — if $p_i$ is set to $T$, then $p_j$ must be (or must not be) set to $T$ in any satisfying truth assignment for the formula. Now construct a dependency graph and the satisfiability problem becomes a graph problem.) □

4. (For CS Students): A Horn formula is a formula in CNF such that in every clause, all except at most one literal is negated. (that is the formula can be written as a conjuction of clauses where, each clause looks like $(p_1 \wedge p_2 \wedge \ldots p_k \rightarrow p_{k+1})$. or like $(p_1 \wedge p_2 \wedge \ldots p_k \rightarrow F)$. Show that there is a polynomial time algorithm for checking whether the forumla is satisfiable in this case. (Note: Such clauses are known as *Horn Clauses*. HORNSAT is in P even as the general formula satisfiability problem is NP complete). □

5. Trace through the steps of the *compactness theorem* to find a satisfying truth assignment (if one exists) given by the proof of the theorem for the collection of formulas: $\{(p_1), (\neg p_1 \vee \neg p_2), (p_1 \vee p_2 \vee p_3), (\neg p_1 \vee \neg p_2 \vee \neg p_3 \vee \neg p_4), \ldots\}$. □

6. (For Math Students:) In this question, we will develop another proof for the compactness theorem for propositional logic. This proof will also make it clear why this theorem is called compactness theorem □

   Consider the set $\{T, F\}$ with the *discrete topology*. Show that w.r.t. this toplogy, every subset is both open and closed and the space is compact. Now look at the product space $\{T, F\}^{\mathcal{N}}$. (i.e., the product space $\mathcal{T} = \{T, F\} \times \{T, F\} \times \ldots$.). What can you say about this space using Tichonoff theorem? Each truth assignment is a point in this product space.

   Now, let $F$ be a collection of formulas. assume that every finite subset of $F$ is satisfiable. Let $p_1, p_2, \ldots$ be the variables in the formulas. Suppose $F'$ be a finite subset of $F$. Denote by $\mathcal{T}_{F'}$ the subset of $\mathcal{T}$ that satisfies $F'$. Denote by $\mathcal{C}$ the collection of all *finite* subsets of $F$. Thus for each $F' \in \mathcal{C}$, $\mathcal{T}_{F'} \neq \emptyset$. Show that for each $F' \in \mathcal{C}$, $\mathcal{T}_{F'}$ is a closed subset of $\mathcal{T}$. Can you conlude that $\bigcap_{F' \in \mathcal{C}} \mathcal{T}_{F'} \neq \emptyset$? (Hint: Tichonoff theorem). Hence argue that any truth assignment in this intersection will satisfy $F$. (Note that this proof extends to the case when there are uncountably many propositional formulas and variables)

7. Let $A$, $B$ be arbitrary sets and $R \subseteq A \times B$ be a binary relation. For any $A' \subseteq A$, $R(A') = \{b' \in B : (a', b') \in R$ for some $a' \in A'\}$. An $R$-Matching from $A$ to $B$ is an injective map $f : A \longrightarrow B$ such that for each $a \in A$, $f(a) \in R(\{a\})$. The relation $R$ is said to satisfy the *Hall Condition* if for each $A' \subset A$, $A'$ **finite**, $|R(A')| \geq |A'|$. □

   The *Hall's theorem* states that if $A$, $B$ are finite sets and $R$ satisfies the Hall's condition, then *there exists* at least one $R$-matching from $A$ to $B$.

   Use compactness theorem to prove that the theorem is true even if $A$ and $B$ are infinite sets.