Logic and Complexity

First order logic

Focus on graphs.

$$\phi = \forall x \, \neg E(x,x) \land \forall x \forall y (E(x,y) \Rightarrow E(y,x)).$$

Interpret this as a statement about graphs as follows:

1. the variables (x and y) range over vertices,

2. E stands for edge relation of graph: $E(x,y)$ is true if and only if there is an edge in the graph from the vertex that x denotes to the vertex that y denotes.

Note: E is not itself the relation but a symbol for it.

First order: variables range only over individual elements of domain of interpretation; *not* over subsets or relations.

Equality: Allow ourselves the normal equality relation symbol $=$.

Interpretation: Given a graph,

- associate relation symbol E with edge relation of graph,
- assign a vertex to each *free* variable.

Models Given

- set Φ of formulae,

- an interpretation over a structure (graph for us).

Structure is a *model* of Φ if each formula of Φ is true under the interpretation.

Sentence: a formula in which each variable is bound by a quantifier.

• Do not need to give an explicit interpretation of individual variables of sentences.

Given: a formula (not necessarily a sentence) ϕ with an interpretation.

Decision problem ϕ-GRAPHS:

INSTANCE: A finite graph G.

QUESTION: Is G a model of ϕ? (i.e., does ϕ express a true property of G?)

Classifies what we can say about finite graphs using first order logic.

THEOREM For any formula ϕ the problem ϕ-GRAPHS is in P.

Question: Does ϕ-GRAPHS capture all of P (on graphs)?

Answer: No (very far from it)!

Let REACHABILITY be

INSTANCE: A directed graph G and two vertices u, v of G.

QUESTION: Is there a (directed) path from u to v in G?

Easy fact: REACHABILITY is in P.

Hard fact: There is no formula ϕ such that REACHABILITY is ϕ-GRAPHS.

• i.e., there is no first order formula $\phi(x, y)$ with two free variables x, y such that for a given graph G and vertices u, v there is a path from u to v if and only if $\phi(u, v)$ is true.

THEOREM [Löwenheim-Skolem Theorem] If a sentence ϕ has models of arbitrarily large size then it has an infinite model.

THEOREM REACHABILITY is not expressible in first order logic.

Directed cycle: is a strongly connected graph such that each vertex has exactly one edge leaving it and exactly one edge entering it

Fact: there are no infinite cycles.

Assume the opposite, i.e., $\phi(x, y)$ exists.

Can now express the three defining properties of a cycle.

1. The graph G is strongly connected:

$$\psi_1 = \forall x \forall y \, \phi(x, y).$$

2. Every vertex has exactly one edge leaving it:

$$\begin{aligned}\psi_2 = & \forall x \exists y \, E(x, y) \wedge \forall x \forall y \forall z ((E(x, y) \wedge E(x, z)) \\ & \Rightarrow y = z).\end{aligned}$$

3. Every vertex has exactly one edge entering it:

$$\begin{aligned}\psi_3 = & \forall x \exists y \, E(y, x) \wedge \forall x \forall y \forall z ((E(y, x) \wedge E(z, x)) \\ & \Rightarrow y = z).\end{aligned}$$

Now take

$$\psi = \psi_1 \wedge \psi_2 \wedge \psi_3.$$

Obviously existence of ψ depends on the assumption that $\phi(x, y)$ exists.

But existence of ψ leads to a contradiction (there are no infinite cycles).

Conclusion: must abandon the assumption that $\phi(x, y)$.