Chapter 4

Posets and Zorn's lemma

Definition 4.1 (Poset). *A partially ordered set or poset is a pair $(X, \leq)$ where X is a set and $\leq$ is a relation on X satisfying:*

1. *Reflexivity: $x \leq x$, $\forall x \in X$.*
2. *Antisymmetry: if $x \leq y$ and $y \leq x$ then $x = y$, $\forall x, y \in X$.*
3. *Transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$, $\forall x, y, z \in X$.*

Write $x < y$ for $x \leq y$ and $x \neq y$. Alternatively in terms of $<$, $\nexists x : x < x$, $x < y$ and $y < z$ implies $x < z$.

Example 4.2. *$(\mathbb{N}, \leq)$, $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are posets (in fact total orders).*

Example 4.3. *$(\mathbb{N}^+, |)$ where ($x|y$ means x divides y) is not a poset.*

Example 4.4. *S a set. $X \subseteq \mathbb{P}(S)$ with $A \leq B$ if $A \subseteq B$.*

Definition 4.5 (Hasse diagram). *A Hasse diagram for a poset is a drawing of the points in the poset with an upwards line from x to y if y covers x (meaning $x < y$ and $\nexists z : x < z < y$).*

Sometimes a Hasse diagram can be drawn for an infinite poset. For example $(\mathbb{N}, \leq)$ but $(\mathbb{Q}, \leq)$ has an empty Hasse diagram.

Definition 4.6 (Chain). *A chain in a poset X is a set $A \subseteq X$ that is totally ordered ($\forall x, y \in A$: have $x \leq y$ or $y \leq x$).*

For example in $(\mathbb{R}, \leq)$ any subset, like $(\mathbb{Q}, \leq)$ is a chain. Note that a chain need not be countable.

Definition 4.7 (Antichain). *An antichain is a subset $A \subseteq X$ in which no two distinct elements are comparable. $\forall x, y : x \neq y$, neither $x \leq y$ nor $y \leq x$.*

Definition 4.8 (Upper bound). *For $S \subseteq X$ and $x \in X$, say x is an upper bound for S if $y \leq x$ $\forall y \in S$.*

Definition 4.9 (Least upper bound, supremum, $\wedge S$). *x is a least upper bound for $S \subseteq X$ if x is an upper bound for S and every upper bound y for S satisfies $x \leq y$.*

Clearly unique if it exists. Write $x = \wedge S = \sup S$ the supremum or join of S.

Definition 4.10 (Complete). *A poset is complete if every set has a supremum.*

Observation 4.11. *Every complete poset X has a greatest element, $\wedge X$ and a least element $\wedge\emptyset$.*

Definition 4.12 (Monotone, order preserving). *A function $f : X \mapsto X$, X a poset, is monotone or order preserving if $x \leq y$ implies $f(x) \leq f(y)$.*

Theorem 4.13 (Knaster-Tarski fixed point theorem). *X a complete poset, $f : X \mapsto X$ order preserving. Then f has a fixed point.*

Proof. Let $E = \{x \in X : x \leq f(x)\}$. Possibly $E = \emptyset$.

Claim. If $x \in E$ then $f(x) \in E$. Proof. $x \leq f(x)$ so $f(x) \leq f(f(x))$ as f order preserving. So $f(x) \in E$.

Let $s = \wedge E$.

Claim. $s \in E$. True if $f(s)$ an upper bound for E (so $s \leq f(s)$). If $x \in E$, $x \leq s$ so $f(x) \leq f(s)$. But $x \in E$ so $x \leq f(x) \leq f(s)$. So $f(s)$ is an upper bound for E.

So $f(s)$ in E by first claim. So $f(s) \leq s$ but second claim showed $s \leq f(s)$ so $f(s) = s$.

□

Corollary 4.14 (Schröder-Bernstein theorem). *A, B have injections $f : A \mapsto B$ and $g : B \mapsto A$ then A, B biject.*

Proof. Want partitions $A = P \cup Q$ and $B = R \cup S$ such that f_p bijects P with R and g_s bijects S with Q.

Then define obvious bijection $h : A \mapsto B$ by taking $h = f$ on P and $h = g^{-1}$ on Q.

Set $P \subseteq A : A \setminus g(B \setminus f(P)) = P$, $R = f(P)$, $S = B \setminus R$, $Q = g(S)$. Consider $(X = \mathbb{P}(A), \subseteq)$. X complete. Define $\theta : X \mapsto X$. $\theta(P) = A \setminus g(B \setminus f(P))$. Then θ is order preserving so it has a fixed point by Knaster-Tarski.

□

Definition 4.15 (Chain-complete). *A (non-empty) poset X is chain-complete if every non-empty chain has a supremum.*

Observation 4.16. *Not all functions on chain-complete posets have fixed points. Any function on an anti-chain is order preserving.*

Observation 4.17. *The non-empty condition is a little pedantic but necessary.*

Definition 4.18 (Inflationary). *$f : X \mapsto X$ is inflationary if $x \leq f(x)$ $\forall x \in X$.*

Not necessarily related to order preserving.

Theorem 4.19 (Bourbaki-Witt theorem). *X is a chain-complete poset, $f : X \mapsto X$ inflationary. Then f has a fixed point.*

Proof. This proof is like battling Godzilla on a tightrope, it has to be carefully choreographed. Although the theorem seems fairly plausible, it has many big consequences.

Fix $x_0 \in X$. Say $A \subseteq X$ closed if

1. $x_0 \in A$
2. $x \in A$ implies $f(x) \in A$
3. C a non-empty chain in A implies $\wedge C \in A$.

Note that any intersection of closed sets is closed.

Let $E = \cap_{A \text{closed}} A$ is closed. Therefore if $A \subseteq E$ then $A = E$.

Assume E is a chain. Let $s = \wedge E$. Then $s \in E$ as E is closed. Therefore $f(s) \in E$. So $f(s) \leq s$. So $f(s) = s$ as f inflationary. So done.

Claim. E is a chain.

Say $x \in E$ is normal if $\forall y \in E : y < x$ then $f(y) \leq x$.

There are two properties of normality we want prove. All $x \in E$ are normal. Secondly, it should satisfy the condition we might naturally describe as "normal": if x normal then $\forall y \in E$ either $y \leq x$ or $y \geq f(x)$.

Once we have done this, we are finished. $\forall x, y \in E$, $y \leq x$ or $y \geq f(x) \geq x$. So E is a chain.

Claim. If x normal then $\forall y \in E$ either $y \leq x$ or $y \geq f(x)$.

Proof of claim. Let $A = \{y \in E : y \leq x \text{ or } y \geq f(x)\}$. Will show A is closed. Any closed subset of E is E so A closed implies $A = E$.

1. $x_0 \in A$. $x_0 \leq x$ ($\forall x \in E$).

2. Given $y \in A$ we need $f(y) \in A$. So have $y \leq x$ or $y \geq f(x)$ and want $f(y) \leq x$ or $f(y) \geq f(x)$.

   If $y < x$ then $f(y) \leq x$ as x is normal.
   If $y = x$ then $f(y) \geq f(x)$.
   If $y \geq f(x)$ then $f(y) \geq y \geq f(x)$.

   So $f(y) \in A$.

3. Given a (non-empty) chain $C \subseteq A$, want $s = \wedge A \in A$.

   If all $y \in C$ have $y \leq x$ then certainly $s \leq x$ because s a supremum. Otherwise some $y \in C$ has $y \geq x$ and not $y \leq x$ so $y \geq f(x)$ as $y \in A$. So $s \geq y \geq f(x)$. So $s \in A$.

So A closed, so A closed subset of smallest possible closed set E so $A = E$.

Claim. Every $x \in E$ is normal.

Proof of claim. Let $N = \{x \in E : x \text{ is normal }\}$. We will show that N is closed so $N = E$.

N is closed:

1. No $y \in E$ has $y < x_0$. So x_0 is normal, $x_0 \in N$.

2. Given x normal want $f(x)$ normal. So must show $y < f(x)$ implies $f(y) \leq f(x)$. By first claim $y < f(x)$ implies $y \leq x$. So $y = x$ or $y < x$. So $f(y) = f(x)$ or $f(y) \leq x \leq f(x)$ (because x is normal).

3. Given a (non-empty) chain $C \subseteq N$ need $s = \wedge C \in N$. That is, we need that if $y < s$ then $f(y) \leq s$ $\forall y \in E$.

   For $y < s$ cannot have $y \geq x$ $\forall x \in C$ (definition of supremum). So some $x \in C$ has not $y \geq x$, so $y < x$ by the first claim. So $f(y) \leq x$ (x normal) so certainly $f(y) \leq s$.

So N closed so $N = E$. So E is a chain.

□

Observation 4.20. *"Now forget the proof" - Dr Leader*

Definition 4.21 (Maximal element of a poset). *Given a poset X an element x is maximal if no $y \in X$ has $y > x$.*

Corollary 4.22 (Every chain-complete poset has a maximal element). *Every chain-complete poset has a maximal element.*

Observation 4.23. *Very non-obvious theorem which trivially implies Bourbaki-Witt (x maximal implies $f(x) = x$).*

Proof. By contradiction. For each $x \in X$ have $\bar{x} \in X$ with $\bar{x} > x$. Then the function $x \mapsto \bar{x}$ is inflationary. So it has a fixed point. Contradiction.

□

Lemma 4.24 (One important chain-complete poset). *Let X be any poset and let P be the collection of all chains of X ordered by inclusion. Then P is chain complete.*

Proof. Let $\{C_i : i \in I\}$ be a chain in P. C_i is a chain in X for all $i \in I$. Note that I need not be countable. Further $\forall i, j \in I$ $C_i \subseteq C_j$ or $C_j \subseteq C_i$.

Now let $C = \cup_{i \in I} C_i$. C is clearly a least upper bound for $\{C_i\}$. We need to show that it is a chain.

Let $x, y \in C$. So $\exists i, j : x \in C_i$ and $y \in C_j$. So $C_i \subseteq C_j$ or $C_j \subseteq C_i$. So x, y related. So C a chain.

□

Corollary 4.25 (Kuratowski's lemma). *Every poset X has a maximal chain.*

Proof. The set of chains of X is a chain-complete poset.

□

Corollary 4.26 (Zorn's lemma). *Let X be a (non-empty) poset in which every chain has an upper bound. Then X has a maximal element.*

Proof. Let C be a maximal chain in X. Let x be an upper bound for C. Then x is maximal. If $y > x$ then $C \cup \{y\}$ is a chain properly containing C. Contradiction.

□

Observation 4.27. *Non-emptiness actually not needed as it follows from the condition that every chain has an upper bound.*

Corollary 4.28 (Every vector space V has a basis). *Every vector space V has a basis.*

Proof. Let $X = \{A \subseteq V : A$ is linearly independent $\}$ ordered by inclusion. We seek the existence of maximal element $A \in X$ using Zorn's lemma. Then we are done because if A does not span V it is not maximal.

1. $\emptyset$ is linearly independent. So $\emptyset \in X$. So $X \neq \emptyset$.

2. Given a chain $\{A_i : i \in I\}$ in X we seek an upper bound S. Let $S = \cup_{i \in I} A_i$. Then $S \supseteq A_i$ $\forall i$ so we just need $S \in X$ (that is, S linearly independent).

   Suppose $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$ for some $x_1, \cdots, x_n \in A$ and $\lambda_1, \cdots, \lambda_n$ not all zero. Have $A_m \in X$ such that A_m contains all the x_i because X is a chain. But this contradicts A_m being linearly independent. So $S \in X$. So every chain has an upper bound.