A Crash Course in Linear Algebra

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1 Definitions

The goal of this section is to provide a brief refresher in the basic terms and concepts of linear algebra, listed here roughly in the order in which they would be covered in an introductory course. Hopefully, you will also find it useful as a reference.

Matrix Product The *matrix product* of an $m \times n$ matrix A and an $n \times p$ matrix B is the $m \times p$ matrix AB whose $(i, j)^{\text{th}}$ element is the dot product of the i^{th} row of A with the j^{th} column of B:

$$\underbrace{\begin{pmatrix} — & A_1 & — \\ — & A_2 & — \\ & \vdots & \\ — & A_m & — \end{pmatrix}}_{m\times n} \underbrace{\begin{pmatrix} | & | & | & | \\ (B^{\text{T}})_1 & (B^{\text{T}})_2 & \dots & (B^{\text{T}})_n \\ | & | & | & | \end{pmatrix}}_{n\times p} = \underbrace{\begin{pmatrix} A_1 \cdot (B^{\text{T}})_1 & A_1 \cdot (B^{\text{T}})_2 & \dots & A_1 \cdot (B^{\text{T}})_n \\ A_2 \cdot (B^{\text{T}})_1 & A_2 \cdot (B^{\text{T}})_2 & \dots & A_2 \cdot (B^{\text{T}})_n \\ \vdots & & \ddots & \vdots \\ A_m \cdot (B^{\text{T}})_1 & A_m \cdot (B^{\text{T}})_2 & \dots & A_m \cdot (B^{\text{T}})_n \end{pmatrix}}_{m\times p}.$$

Here, A_i is the i^{th} row of A and $(B^T)_j$ is the j^{th} column of B (i.e., the j^{th} row of the transpose of B). More compactly,

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

A special case of matrix multiplication is the matrix-vector product. The product of an $m \times n$ matrix A with an n-element column vector x is the m-element column vector Ax whose i^{th} entry is the dot product of the i^{th} row of A with x.

Vector Space A *vector space* is any set V for which two operations are defined:

- *Vector Addition*: Any two vectors v_1 and v_2 in V can be added to produce a third vector $v_1 + v_2$ which is also in V.
- *Scalar Multiplication*: Any vector v in V can be multiplied ("scaled") by a real number[1] $c \in \mathbb{R}$ to produce a second vector cv which is also in V.

and which satisfies the following axioms:

1. Vector addition is commutative: $v_1 + v_2 = v_2 + v_1$.
2. Vector addition is associative: $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$.
3. There exists an additive identity element 0 in V such that, for any $v \in V$, $v + 0 = v$.
4. There exists for each $v \in V$ an additive inverse $-v$ such that $v + (-v) = 0$.
5. Scalar multiplication is associative: $c(dv) = (cd)v$ for $c, d \in \mathbb{R}$ and $v \in V$.
6. Scalar multiplication distributes over vector and scalar addition: for $c, d \in \mathbb{R}$ and $v_1, v_2 \in V$, $c(v_1 + v_2) = cv_1 + cv_2$ and $(c+d)v_1 = cv_1 + cv_2$.
7. Scalar multiplication is defined such that $1v = v$ for all $v \in V$.

Any element of such a set is called a *vector*; this is the rigorous definition of the term.

Function Space A *function space* is a set of functions that satisfy the above axioms and hence form a vector space. That is, each vector in the space is a function, and vector addition is typically defined in the obvious way: for any functions f and g in the space, their sum $(f + g)$ is defined as $(f + g)(x) \equiv f(x) + g(x)$. Common function spaces are $\mathbb{P}^n$, the space of n^{th}-degree polynomials, and C^n, the space of n-times continuously differentiable functions.

Inner Product An *inner product* $\langle \cdot, \cdot \rangle$ on a real vector space V is a map that takes any two elements of V to a real number. Additionally, it satisfies the following axioms:

1. It is *symmetric*: for any $v_1, v_2 \in V$, $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$.
2. It is *bilinear*: for any $v_1, v_2, v_3 \in V$ and $a, b \in \mathbb{R}$, $\langle av_1 + bv_2, v_3 \rangle = a\langle v_1, v_3 \rangle + b\langle v_2, v_3 \rangle$ and $\langle v_1, av_2 + bv_3 \rangle = a\langle v_1, v_2 \rangle + b\langle v_1, v_3 \rangle$.
3. It is *positive definite*: for any $v \in V$, $\langle v, v \rangle \geq 0$, where the equality holds only for the case $v = 0$.

[1] More general definitions of a vector space are possible by allowing scalar multiplication to be defined with respect to any arbitrary field, but the most common vector spaces take scalars to be real or complex numbers.

These axioms can be generalized slightly to include complex vector spaces. An inner product on such a space satisfies the following axioms:

1. It is *conjugate symmetric*: for any $v_1, v_2 \in V$, $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$, where the overbar denotes the complex conjugate of the expression below it.
2. It is *sesquilinear* (linear in the first argument and conjugate-linear in the second): for any $v_1, v_2, v_3 \in V$ and $a, b \in \mathbb{C}$, $\langle av_1 + bv_2, v_3 \rangle = a\langle v_1, v_3 \rangle + b\langle v_2, v_3 \rangle$ and $\langle v_1, av_2 + bv_3 \rangle = \overline{a}\langle v_1, v_2 \rangle + \overline{b}\langle v_1, v_3 \rangle$.
3. It is *positive definite*, as defined above.

The most common inner product on $\mathbb{R}^n$ is the dot product, defined as

$$\langle u, v \rangle \equiv \sum_{i=1}^{n} u_i v_i \quad \text{for} \quad u, v \in \mathbb{R}^n.$$

Similarly, a common inner product on $\mathbb{C}^n$ is defined as

$$\langle u, v \rangle \equiv \sum_{i=1}^{n} u_i \overline{v_i} \quad \text{for} \quad u, v \in \mathbb{C}^n$$

Note, however, that in physics it is often the first vector in the angled brackets whose complex conjugate is used, and the second axiom above is modified accordingly. In function spaces, the most common inner products are integrals. For example, the $\mathcal{L}^2$ norm on the space $C^n[a, b]$ of n-times continuously differentiable functions on the interval $[a, b]$ is defined as

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

for all $f, g \in C^n[a, b]$.

Linear Combination A *linear combination* of k vectors, $v_1, v_2, \ldots, v_k$, is the vector sum

$$S = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \tag{1}$$

for some set of scalars $\{c_i\}$.

Linear Independence A set of vectors is *linearly independent* if no vector in it can be written as a linear combination of the others. Equivalently, the vectors $v_1, v_2, \ldots, v_n$ are linearly independent if the only solution to

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

is $c_1 = c_2 = \cdots = c_n = 0$.

For example, the vectors $(1, 1)$ and $(1, -1)$ are linearly independent, since there is no way to write $(1, 1)$ as a scalar multiple of $(1, -1)$. The vectors $(1, 1)$, $(1, -1)$, and $(1, 0)$ are not linearly independent, however, since any one can be written as a linear combination of the other two, as in $(1, -1) = 2 * (1, 0) - (1, 1)$.

Span The *span* of a set of vectors $\{v_1, v_2, \ldots, v_k\}$, denoted $\text{span}(v_1, v_2, \ldots, v_k)$ is the set of all possible linear combinations of the vectors. Intuitively, the span of a collection of vectors is the set of all points that are "reachable" by some linear combination of the vectors. For example, the vectors $v_1 = (1, 0, 0)$, $v_2 = (0, -2, 0)$, and $v_3 = (0, 5, 5)$ span $\mathbb{R}^3$, since any three-component vector can be written as a sum of these three vectors using the proper coefficients. In contrast, the vectors v_1, v_2, and $v_4 = (1, 1, 0)$ span only the x-y plane and not all of $\mathbb{R}^3$, since none of these vectors has a component along the z direction.

Basis and Dimension A *basis* of a vector space is any set of vectors which are linearly independent and which span the space. The vectors v_1, v_2, and v_3 in the previous example form a basis for $\mathbb{R}^3$, while the vectors v_1, v_2, and v_4 do not form a basis of either $\mathbb{R}^3$ (they do not span the space) or $\mathbb{R}^2$ (they are not linearly independent). There are infinitely many bases for any given space (except for the trivial space, consisting only of the number 0), and all of them contain the same number of basis vectors. This number is called the *dimension* of the space.

Range/Image/Column Space The *range* of a matrix, also known as its *image* or *column space*, is the space spanned by its columns. Equivalently, it is the set of all possible linear combinations of the columns of the matrix. For example, if one views an $m \times n$ matrix as a linear transformation operating on $\mathbb{R}^n$, then the range of the matrix is the subspace of $\mathbb{R}^m$ into which it maps $\mathbb{R}^n$.

Rank The *rank* of a matrix is the dimension of its column space (or that of its row space, since it turns out that these dimensions are equal). Equivalently, it is the number of linearly independent columns (or rows) in the matrix.

Kernel/Null Space The *kernel* or *null space* of an $m \times n$ matrix A, denoted $\ker(A)$, is the set of all vectors that the matrix maps to the zero vector. In symbols,

$$\ker(A) = \{x \in \mathbb{R}^n | Ax = 0\}.$$

The following terms relate only to square matrices. They are arranged roughly in order of conceptual importance.

Eigenvalues and Eigenvectors An *eigenvector* of a matrix A is a vector v which satisfies the equation

$$Av = \lambda v$$

for some complex number λ, which is called the corresponding *eigenvalue*. Note that λ might have a nonzero imaginary part even if A is real. (See below for more information about eigenvalues and eigenvectors.)

Invertible An *invertible* matrix is any matrix A for which there exists a matrix B such that $AB = BA = I$, where I is the identity matrix of the same dimension as A and B. The matrix B is said to be the *inverse* of A, and is usually denoted A^{-1}: $AA^{-1} = A^{-1}A = I$. A matrix which is not invertible is called *singular*. Invertible and singular matrices can be distinguished by their determinants:

$$\det(A) = 0 \iff A \text{ is singular}$$
$$\det(A) \neq 0 \iff A \text{ is invertible}$$

Diagonalizable A *diagonalizable* matrix A is any matrix for which there exists an invertible matrix S such that $S^{-1}AS = D$, where D is a diagonal matrix (i.e. all of its off-diagonal elements are zero). A square matrix which is not diagonalizable is called *defective*.

Orthogonal (Unitary) An *orthogonal* matrix is any square matrix A with real elements that satisfies $A^T = A^{-1}$, so $A^TA = AA^T = I$. Equivalently, a real, square matrix is orthogonal if its columns are orthonormal with respect to the dot product. A *unitary* matrix is the complex equivalent; a complex, square matrix is unitary if it satisfies $A^\dagger = A^{-1}$ (where $A^\dagger \equiv \overline{A^T}$), so $A^\dagger A = AA^\dagger = I$.

Symmetric (Hermitian) A *symmetric* matrix is any real matrix that satisfies $A^T = A$. Similarly, a *Hermitian* matrix is any complex matrix that satisfies $A^\dagger = A$.

Normal A *normal* matrix is any matrix for which $A^TA = AA^T$ (or $A^\dagger A = AA^\dagger$ for complex matrices). For example, all symmetric and orthogonal matrices are normal.

Positive and Negative Definite and Semidefinite A *positive definite* matrix is any symmetric or Hermitian matrix A for which the quantity $\overline{v^T}Av \geq 0$ for all v, with the equality holding only for the case $v = 0$. If the inequality holds but there is at least one nonzero vector v for which $\overline{v^T}Av = 0$, then the matrix is called *positive semidefinite*. Likewise, a *negative definite* matrix is any symmetric or Hermitian matrix A for which $\overline{v^T}Av \leq 0$ for all v, with the equality holding only for $v = 0$. If $\overline{v^T}Av \leq 0$ for all v but $\overline{v^T}Av = 0$ for at least one nonzero v, the matrix A is called *negative semidefinite*.

2 Important Concepts

This section contains brief explanations of the two most important equations of linear algebra: $Ax = b$ and $Av = \lambda v$. If you haven't had a full introductory course in linear algebra, you might not have seen these interpretations before. Hopefully, you'll find them intuitive and helpful!

2.1 The Meaning of $Ax = b$

One of the most common applications of linear algebra is to solve the simple matrix equation $Ax = b$. In this equation, A is a given $m \times n$ matrix, b is a given vector in $\mathbb{R}^m$, and the problem is to solve for the unknown vector x in $\mathbb{R}^n$. This equation arises wherever one must solve m linear equations for n unknowns.

Notice that the matrix-vector product Ax on the left-hand side is nothing other than a linear combination of the columns of A with coefficients given by the elements of x:

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n \end{pmatrix}$$
$$= x_1 \begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{pmatrix} + x_2 \begin{pmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{pmatrix}.$$

That is, the question of whether or not $Ax = b$ has a solution is essentially the question of whether or not the vector b lies in the space spanned by the columns of A (remember, this is called the *range* of A). There are three possibilities:

1. $b \notin \text{rng}(A) \rightarrow$ there is no x for which $Ax = b$

2. $b \in \text{rng}(A)$ and the columns of A are linearly independent $\rightarrow$ there is one and only one x that solves $Ax = b$

3. $b \in \text{rng}(A)$ and the columns of A are not linearly independent $\rightarrow$ there are infinitely many vectors x that solve $Ax = b$.

In case (1), no matter how the columns of A are weighted, they cannot sum to give the vector b. In case (2), the fact that b lies within the range of A guarantees that there is at least one solution (problem 3.3 asks you to show that it is unique). Finally, in case (3), not only does b lie in the range of A, guaranteeing that a solution exists, but at least one of the columns of A can be written as a linear combination of the others. As a result, there are infinitely many linear combinations of them which sum to give b.

The conceptual power of this interpretation is that it lends a *geometric* significance to the *algebraic* equation $Ax = b$. You can picture the n columns of A as vectors in $\mathbb{R}^m$. Together, they span some space, which could be all of $\mathbb{R}^m$ or only some proper subspace. In the former case, the vectors can be combined, with an appropriate choice of scalar coefficients, to produce any vector in $\mathbb{R}^m$. In the latter case, in contrast, there are some vectors in $\mathbb{R}^m$ that lie outside of the span of the columns of A. If the vector b in $Ax = b$ happens to be one of those vectors, then no possible linear combination of the columns of A can reach it, and the equation $Ax = b$ has no solution.

2.2 The Eigenvalue Equation, $Av = \lambda v$

Another ubiquitous and fundamentally important equation in linear algebra is the relation $Av = \lambda v$, where A is an $n \times n$ matrix (notice that it must be square in order for the left- and right-hand sides to have the same dimension), v is an n-element vector, and λ is a constant. The task is to solve for all eigenvectors v and corresponding eigenvalues λ that satisfy this relation.

Rearranging the eigenvalue equation gives

$$(A - \lambda I)v = 0, \tag{2}$$

where I is the $n \times n$ identity matrix. The only way for this equation to have a nontrivial solution is for the matrix $(A - \lambda I)$ to be singular, in which case its determinant is zero. This fact provides a useful method for finding the eigenvalues of small matrices. For example, to find the eigenvalues of

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix},$$

first construct the matrix $(A - \lambda I)$ and then require that its determinant vanish:

$$\begin{aligned} \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 9 = 0 \\ \rightarrow \quad (\lambda + 2)(\lambda - 4) = 0 \end{aligned}$$

The eigenvalues of A are therefore $\lambda_1 = -2$ and $\lambda_2 = 4$. To find the eigenvectors, substitute these numbers into (2). For example,

$$\begin{aligned} &\begin{pmatrix} 1-\lambda_1 & 3 \\ 3 & 1-\lambda_1 \end{pmatrix} v_1 = 0 \\ &\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\rightarrow \quad v_1^{(2)} = -v_1^{(1)} \quad \rightarrow \quad v_1 = v_1^{(1)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

where $v_1^{(1)}$ is an arbitrary scalar (note that $A(cv_1) = \lambda_1(cv_1)$, so eigenvectors are only unique up to a multiplicative constant).

But what does it mean for a vector to be an eigenvector of a matrix? Is there a geometric interpretation of this relationship similar to that given for $Ax = b$ in the previous section? It turns out that there is. It is helpful to think of square matrices as operators on vector spaces, since any linear transformation on $\mathbb{R}^n$ (or $\mathbb{C}^n$) can be represented by an $n \times n$ matrix. Examples of such transformations include rotations, reflections, shear deformations, and inversion through the origin. The eigenvectors of such a matrix, then, are special directions in that space that are unchanged by the transformation that the matrix encodes. A vector that points in one of these directions is scaled by a constant factor (the eigenvalue, λ) upon multiplication by the matrix, but it points along the same axis as it did before the matrix operated on it.

This geometric interpretation is especially useful when zero is an eigenvalue of a matrix. In that case, there is at least one vector (there can be more) that, when multiplied by the matrix, maps to the zero vector. Let A be such a matrix, and let v_0 be an eigenvector of A whose eigenvalue is zero. Then, $Av_0 = 0v_0 = 0$. Likewise, any scalar multiple cv_0 also maps to zero: $Acv_0 = cAv_0 = c0 = 0$. As a result, an entire axis through $\mathbb{R}^n$ (here A is taken to be $n \times n$ and real, as an example) collapses to the zero vector under the transformation represented by A, so the range of A is a lower-dimensional subspace of $\mathbb{R}^n$.

To see why this is so, consider a 3×3 real matrix A with one eigenvector v_0 whose eigenvalue is zero. In this case there is a line in $\mathbb{R}^3$—consisting of all multiples of v_0—that collapses to zero under multiplication by A. Choose as a basis of $\mathbb{R}^3$ the vector v_0 and any two additional linearly independent vectors, v_1 and v_2. Then, since this set forms a basis for the whole space, any vector z in $\mathbb{R}^3$ can be written as a linear combination of these three:

$$z = c_0 v_0 + c_1 v_1 + c_2 v_2$$

The range of A is then the set $\{Az\}$, where z is varied over all of $\mathbb{R}^3$. That is,

$$\begin{aligned}
\text{rng}A &= \{Az \mid z \in \mathbb{R}^3\} \\
&= \{c_0 Av_0 + c_1 Av_1 + c_2 Av_2 \mid c_0, c_1, c_2 \in \mathbb{R}\} \\
&= \{c_1 Av_1 + c_2 Av_2 \mid c_1, c_2 \in \mathbb{R}\}
\end{aligned}$$

The vectors Av_1 and Av_2 evidently form a basis of the range of A, which must therefore be two-dimensional. Likewise, had there been another linearly independent vector that A mapped to the zero vector, we could have chosen that one as v_1. In that case, only v_2 would span the range of A, which would then be only one-dimensional.

More generally, if there are at most k linearly independent vectors $v_1, v_2, \ldots v_k$ that an $(n \times n)$ matrix A maps to the zero vector, then we can choose as a basis of $\mathbb{R}^n$ these k vectors and $n-k$ additional linearly independent vectors. Writing every vector in $\mathbb{R}^n$ as a linear combination of these n vectors, we see that, under multiplication by A, only $n-k$ degrees of freedom survive, so the resulting space, which is the range of A, is only $(n-k)$-dimensional.

This result relates directly back to the geometric interpretation of $Ax = b$ described in (2.1). When a matrix has at least one eigenvector with zero eigenvalue, its range is a lower-dimensional subspace of its domain. As a result, any vector b that lies outside of this subspace lies outside the range of A, so in such a case there is no vector x that solves $Ax = b$. This is why a matrix is singular if one of its eigenvalues is zero; if it were invertible (i.e. not singular), there would always exist such an x (equal to $A^{-1}b$) for every b in the domain of A. Alternatively, we could observe that, since there are many vectors (indeed, an entire subspace!) that get mapped to the zero vector, the mapping defined by A is necessarily many-to-one, so the whole concept of an inverse operation does not make sense in this case.

3 Exercises

3.1 Which of the following sets are vector spaces?

(a) $\mathbb{C}^3$, the set of all ordered triples of complex numbers, with scalar multiplication defined over the complex numbers

(b) $\mathbb{C}^3$, with scalar multiplication defined over the real numbers

(c) $\mathbb{R}^3$, the set of all ordered triples of real numbers, with scalar multiplication defined over the complex numbers

(d) The subset of $\mathbb{R}^2$ enclosed by the unit circle, with scalar multiplication defined over the real numbers

(e) The line $y = 4x$ (i.e. the subset of $\mathbb{R}^2$ comprised by the points on this line), with scalar multiplication over $\mathbb{R}$

(f) The line $y = 4x + 2$, with scalar multiplication over $\mathbb{R}$

(g) The subset of $\mathbb{R}^3$ bounded above by the plane z = 10, with scalar multiplication over $\mathbb{R}$

(h) The functions $f(x) = x^3$, $g(x) = \cos(\sqrt{x})$, and $h(x) = 1$, where $x \in [0, 1]$, and all linear combinations of these functions with real coefficients, with function addition and scalar multiplication defined as usual

(i) The set of all solutions $f(x)$ of the differential equation $f' + f = 0$

3.2 Give an example of a basis for each of the following vector spaces. What is the dimension of each?

(a) $\mathbb{C}^4$, the space of all ordered quadruples of complex numbers, with scalar multiplication defined over the complex numbers

(b) $\mathbb{P}^3$, the space of all polynomials of degree less than or equal to three that have real coefficients

(c) $\mathcal{M}^{2\times 2}$, the space of all 2×2 matrices with real elements

(d) The set of all solutions $f(x)$ of the differential equation $f'' + \omega^2 f = 0$

3.3 Let x be a solution to $Ax = b$, and let the columns of A be linearly independent. Prove that x is the *only* solution.

Hint: Just multiplying by A^{-1} won't do—you still need to prove that the inverse exists and is unique! A better way is to prove this statement by contradiction: assume that x is *not* a unique solution and proceed until you contradict one of the given facts.

3.4 Find the eigenvalues and eigenvectors of the following matrices.

(a)

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

3.5 Prove the following statements about eigenvalues and eigenvectors:

(a) If λ is a complex eigenvalue of a real matrix A and v is the corresponding eigenvector, then $\overline{\lambda}$ is also an eigenvalue of A with eigenvector $\overline{v}$.

(b) If λ is an eigenvalue of A, then it is also an eigenvalue of $P^{-1}AP$ for any invertible matrix P.

(c) The eigenvalues of a real symmetric matrix are real. (*Hint:* $(AB)^T = B^T A^T$)

(d) Eigenvectors of a symmetric matrix that belong to distinct eigenvalues are orthogonal.

3.6 Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.$$

Are these vectors linearly independent? What is the dimension of the space they span? Use two methods to answer these questions:

(a) Recall that if these vectors are linearly independent, then none can be written as a linear combination of the others. As a direct result, the only solution to the equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \tag{3}$$

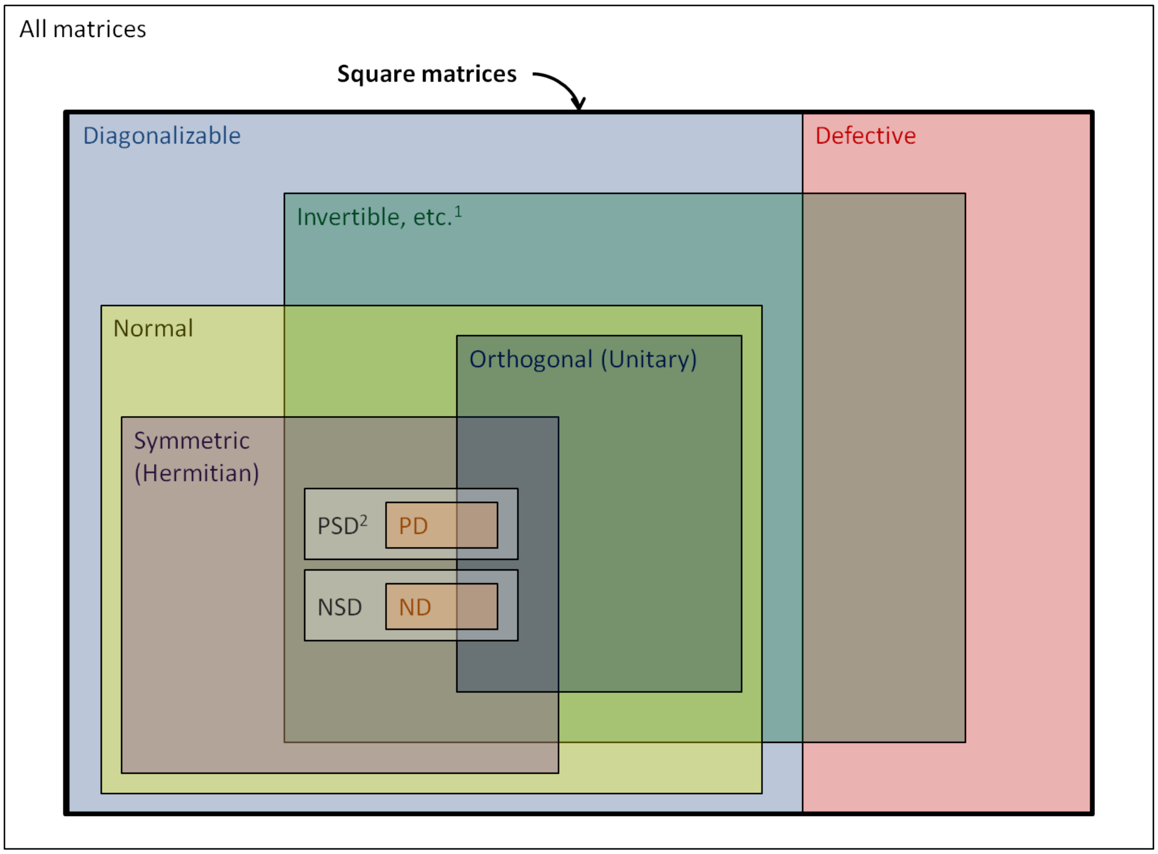
is the trivial solution, $c_1 = c_2 = c_3 = 0$. Write (3) in matrix form and solve it using substitution, row reduction, or MATLAB (make sure you are at least familiar with the other methods before you use MATLAB, though). Is the trivial solution the only solution?

(b) Notice that the matrix that you constructed in (3.6a) is square. Find its eigenvalues. What do they tell you about the vectors v_1, v_2, and v_3? (*Hint:* What is the null space of this matrix?)

3.7 This crash course doesn't include a description of how to row-reduce a matrix, but it's a handy skill to have! Read up on it from another source, then put the following system of equations into matrix form and solve for x, y, and z.

$$\begin{aligned} 3x - y + 2z &= -3 \\ 6x - 2y + z &= -3 \\ -2y - 5z &= -1 \end{aligned}$$

4 Appendix: Venn Diagram of Matrix Types



1. The following statements about a matrix A are equivalent:
   (a) A is invertible.
   (b) The equation $Ax = b$ always has a unique solution.
   (c) Zero is not an eigenvalue of A.
   (d) $\det A \neq 0$
   (e) A is square and has full rank.
2. PSD: Positive Semidefinite
   PD: Positive Definite
   NSD: Negative Semidefinite
   ND: Negative Definite