## Assignment III

1. Let $G$, $H$ be (Abelian) groups. A function $f : G \longrightarrow H$ is a group homomorphism if $f$ satisfies for all $a, b \in G$, $f(a+b) = f(a) + f(b)$. (The difference between a homomorphism and isomorphism is that the condition of bijectivity is dropped. Thus, a homomorphism preserves the group operation, but can be "lossy"). Define $ker(f) = \{a \in G : f(a) = 0\}$. (The kernel, also called null space, is the set of elements in $G$ whose map is 0). Define $Img(f) = \{f(a) : a \in G\}$. ($Img(f)$ could be a proper subset of $H$ when $f$ is not surjective.) In each of the following maps, find $ker(f)$ and $img(f)$. ☐
   a) $f((x, y) = x + y$ from $(\mathbf{R^2}, +)$ to $(\mathbf{R}, +)$. b) $f(x) = x \mod n$ from $\mathbf{Z}$ to $\mathbf{Z_n}$.
   c) $f(x) = x \mod 5$ from $\mathbf{Z_{10}}$ to $\mathbf{Z_5}$.

2. If $f$ is a homomorphism from a (Abelian) group $G$ to a group $H$, show that $ker(f)$ is a subgroup of $G$ and $Img(f)$ is a subgroup of $H$. (Comment: Thus, $|ker(f)|$ must divide $|G|$ and $|Img(f)|$ must divide $|H|$ when $G, H$ are finite). ☐

3. If $m, n$ are positive integers with $m < n$, Show that the map $f(x) = x \mod m$ from $\mathbf{Z_n}$ to $\mathbf{Z_m}$ is a group homomorphism if and only if $m$ divides $n$. (Hint: Suppose $n = qm + r$, use the fact that $f(\sum_1^n 1) = f(0) = 0$ and the fact that $f(m) = 0$ by definition). ☐

4. Let $f$ be a homomorphism from a (Abelian) group $G$ to a (Abelian) group $H$. Let $S = ker(f)$. ☐
   1. Show that $f(a) = f(b)$ if and only if $a - b \in S$.
   2. let $a \in G$. Consider the coset $a + S$. Show that for any $x \in G$, show that $f(x) = f(a)$ if and only if $x \in a + S$. (Why does this follow from the previous question immedietely?). Thus, each coset of $S$ in $G$ gets mapped exactly to the same point in $Img(f)$ and conversely points that gets mapped to the same point in the image must belong to the same coset.
   3. For the $f((x, y) = x + y$ from $(\mathbf{R^2}, +)$ to $(\mathbf{R}, +)$, find the equation to set of points $(x, y)$ in $(\mathbf{R^2}, +)$ whose image is the same as $f(1, 2)$.
   4. For $f(x) = x \mod 5$ from $\mathbf{Z_{10}}$ to $\mathbf{Z_5}$, find $S$ and all the cosets of $S$. Identify the image point to which each coset is mapped to.

   The observations in the previous question shows that we can one to one map points in $Img(f)$ with cosets of $S$ in $G$. The next question develops this correspondance formally.

5. In the last Question of Assignment II, it was asked to prove that if $S$ is any subgroup of an Abelian group $G$, we can define addition of cosets by the rule $(a+S)+(b+S) = (a+b)+S$. The question asked you to show that with this definition of addtion, the set of cosets of $G$ with respect to $S$ forms a group. This group is called the **quotient group** of $G$ defined by $S$, denoted by $G/S$. Note that each element in $G/S$ is a coset of the form $a + S$. Also note that $S$ is the identity element in $G/S$ and $(-a) + S$ is the inverse of the coset $a + S$ in $G/S$. Let $f$ be a homomorphism from a group $G$ to a group $H$. Let $S = ker(f)$ ☐
   1. Define the map $\Phi : G/S \longrightarrow Img(f)$ as follows: $\Phi(a+S) = f(a)$. (The map simply associates the coset $a + S$ in $G/H$ to the element $f(a)$ in $Img(f)$).
   2. Show that $\Phi((a + S) + (b + S)) = \Phi(a + S) + \Phi(b + S) = f(a) + f(b)$.
   3. Show that the map is injective ($f(a + S) = 0$ if and only if $a + S = S$). Note that since $\Phi$ is surjective by definition. Hence, $\Phi$ is bijective and hence an isomorphism in view of the previous question. This observation is called the **first homomorphism theorem** of groups.
   4. Consider the homomorphism $f(x) = x \mod 4$ from $\mathbf{Z}$ to $\mathbf{Z_4}$. In this case $S = 4\mathbf{Z}$ (The general equation to $S$ is called the "complimentary function,".) Find the "general solution" for $f(x) = 3$.

   [Note: In view of the homomorphism theorem, when $f$ is a homomorphism from a group $G$ to a group $H$, given $b \in Img(f)$, in order to solve $f(x) = b$, we must find any one solution $x_0$ (commonly called a "particular solution") and find the coset $x_0 + S$ defined by this particular solution. For this it suffices to add the general equation for $S$ ("complimentary function") to the "particular solution" $x_0$. This method is commonly used to solve differential equations. ]