Assignment V

1. Suppose a vector $v \in \mathbf{R^3}$ has coordinates $(1, 1, 1)$ w.r.t the basis $(1, 1, 0), (1, 0, 1), (0, 1, 1)$, Find its coordinates with respect to the basis $(1, 0, 0), (1, 1, 0), (1, 1, 1)$. Find the matrix of basis translation between these two bases. □

2. In all the questions below, V is a vector space over a field F of dimension n unless stated otherwise. Let U be a subspace of V of dimension k. Consider the cosets defined by U in V. (remember the work in all previous assignments!) Let $v_1 + U$ and $v_2 + U$ be cosets. We can define addition of cosets in a natural way as $(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U$. Since vectors form a group with respect to addition, we have already shown that this operation is well defined. Define scalar multiplication of cosets also in the natural way $\alpha(v + U) = (\alpha v) + U$, for $\alpha \in F$ and $v \in V$. Show that if $v' = \alpha v$, then $v + U = v' + U$, establishing that multiplication defined this way also is well defined. With these operations, show that the set of cosets form a vector space. This space is called the *quotient space* defined by U in the vector space V denoted by V/U. □

3. In the previous question, Suppose $u_1, u_2, ..., u_k$ is a basis of U. Add vectors outside U to extend this set to a basis $u_1, u_2, ..., u_k, b_1, b_2, ..., b_r$ of V. (Clearly $r = n - k$ (why?); how do you do this extension systematically?). Show that in the quotient space V/U, $b_1 + U, b_2 + U, ..., b_r + U$ are linearly independent. Show that they form a basis of V/U. Hence conclude that $dim(V/U) = dim(V) - dim(U)$. □

4. In $V = \mathbf{R^3}$, consider the subspace U defined by the plane $x + y + z = 0$. what is dimension of V/U? Find a vector $v \in V$ such that $u + W$ is a basis of V/U. Repeat the same question with U being replaced by the line defined by the intersection of the planes $x + y + z = 0$ and $z = 0$. □

5. Consider the linear map from $\mathbf{R^3}$ to $\mathbf{R^2}$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$. Find the matrix of this linear transformation if the basis on either side is the standard basis. What is the matrix of the map if the basis in $\mathbf{R^2}$ is changed to $(1, 1), (1, -1)$? What is the matrix when the basis for $\mathbf{R^3}$ is changed to $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ and the basis of $\mathbf{R^2}$ is the standard basis? What is the matrix of the map if the basis for $\mathbf{R^3}$ is $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ and basis for $\mathbf{R^2}$ is $(1, 1), (1, -1)$? □

6. In the above question, find the equations to $img(T)$ and $ker(T)$ (assume standard basis on both sides). Find a basis for $ker(T)$ and $Img(T)$. Find a collection of vectors in $\mathbf{R^3}$ whose images under T form a basis for $img(T)$. What is $Rank(T)$, $Nullity(T)$? □

7. Let T be a linear transformation from V to W over a field F. Let $Dim(V) = n, U = Nullspace(T)$. Let $Nullity(T) = k$ and $rank(T) = r$. Let $u_1, u_2, .., u_k$ be a basis of U. Suppose we extend this set with vectors $b_1, b_2, .., b_r$ to form a basis of V, is it always true that $T(b_1), T(b_2), .., T(b_r)$ forms a basis of $Img(T)$? Give a proof/counterexample. □

8. In the above question, Define the map $f : V/U \longrightarrow Img(T)$ as follows: $f(v + U) = T(v)$. □

    1. Show that f is well defined. That is, if $v + U = v' + U$ then $f(v + U) = f(v' + U)$.
    2. Show that f is a linear map from V/U to W.
    3. Show that f is injective; Hence conclude that f is an isomorphism between V/W and $Img(T)$. (This observation is called the first homomorphism theorem for vector spaces.) Conclude that $Dim(Img(T)) = Dim(V/W) = n - k$ (the last equality following from the the third question). This gives another proof for the Rank Nullity Theorem.