Assignment VI

1. Consider the vector space $\mathbf{R^3}$. Consider the space W defined by the equation $x + y + z = 0$. Recall that W^* consists of all linear functions l from $\mathbf{R^3}$ to $\mathbf{R}$ such that $l(w) = 0$ for all $w \in W$. Find a basis of W^*.

2. Find the dual basis in $\mathbf{R^3}$ corresponding to the basis $[1, 0, 0]^T, [1, 1, 0]^T, [1, 1, 1]^T$.

3. Find a basis of Eigen vectors for the operator T on $\mathbf{R^2}$ defined by $T(e_1) = e_1 - e_2$ and $T(e_2) = e_2 - e_1$. Find the matrix of T with respect to this basis.

4. Show that a linear operator T is not a bijective map if and only if 0 is an Eigen value.

5. Suppose T is a linear operator on a vector space V of dimension n over a field F. Suppose b_1 and b_2 are Eigen vectors of T with Eigen values λ_1 and λ_2, with $\lambda_1 \neq \lambda_2$. Show that b_1 and b_2 are linearly independent. Extend this argument to show that if $b_1, b_2, .., b_n$ are Eigen vectors of T corresponding to *distinct* Eigen values $\lambda_1, \lambda_2, ..., \lambda_n$, then $b_1, b_2, .., b_n$ are linearly independent. From this, conclude that if T has n distinct Eigen values, then T is diagonalizable.

6. An n bit binary linear code C is a linear subspace of $\mathbf{F_2^n}$. If $dim(C) = k$, then we say C is a (n, k) linear code. A $k \times n$ matrix whose rows are linearly independent and spans C is called a *generator matrix* for C. A generator matrix for the complement space of C (denoted by C^0, consists of all vectors v in F_2^n satisfying $v^T x = 0$ for all $x \in C$) is called a *parity check matrix* for C. Prove that the parity check matrix for C must be an $(n - k) \times n$ matrix. For the code $C = \{0110, 1111, 0000, 1001\}$ in F_2^4, find a generator matrix and a parity check matrix. Note that C^0 itself is a linear code and is called the *dual code* of C.

7. Suppose U and W are subspaces of a vector space V such that $U \cap W = \{0\}$. Define $U \oplus W = \{u + w : u \in U, w \in W\}$. Show that $U + W$ is a subspace of V with $dim(U \oplus W) = dim(U) + dim(W)$. (Show that if $u_1, u_2, ..., u_l$ and $w_1, w_2, ..., w_k$ are bases for U and W then $u_1, u_2, ..., u_l, w_1, w_2, ..., w_k$ is a basis for $U \oplus W$). In general, show that if $U_1, U_2, .., U_k$ are subspaces of V such that $U_i \cap U_k = \emptyset$, the $U_1 \oplus U_2 \oplus ... \oplus U_k$ is a subspace of V with dimension $dim(U_1) + dim(U_2) + ... + dim(U_k)$.

8. Suppose T is a linear operator on an n dimensional vector space V. Let $\lambda_1, \lambda_2, ..., \lambda_k$ be the distinct Eigen values of V. Define $E_{\lambda_i} = \{v \in V : Tv = \lambda_i v\}$. E_{λ_i} is called the Eigen space associated with the Eigen value λ_i. $dim(E_{\lambda_i})$ is called the **geometric multiplicity** of the Eigen value λ_i. Show that for each λ_i, E_{λ_i} is a subspace of V. If $i \neq j$, then show that $E_{\lambda_i} \cap E_{\lambda_j} = \emptyset$.

9. Find the Eigen spaces associated with all the Eigen vectors of the matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

   Suppose T is a linear operator on a vector space V. Let $\lambda_1, \lambda_2, ..., \lambda_k$ be the Eigen values of T and let $E_{\lambda_1}, E_{\lambda_2}, ..., E_{\lambda_k}$ be the Eigen spaces associated with these Eigen values. Suppose $dim(E_{\lambda_1}) + dim(E_{\lambda_2}) + ... + dim(E_{\lambda_k}) = dim(V)$. Then show that T is diagonalizable. In particular, if $b_1^1, b_2^1, ...$ forms a basis of E_{λ_1}, $b_1^2, b_2^2, ...$ forms a basis for E_{λ_2} etc, then show that all these basis vectors together constitute a digonalizing basis for V.

10. Find a basis that diagonalizes the matrix in $\mathbf{R^2}$, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$