Assignment VII

1. Consider the vector space $\mathbf{C^4}$. Show that the basis defined by $b_1 = [1, 1, 1, 1]^T$, $b_2 = [1, -1, 1, -1]^T$, $b_3 = [1, i, -1, -i]^T$, $b_4 = [1, -i, 1, i]^T$ is an orthogonal (but not orthonormal) basis. Find the matrix of translation from the standard basis to this basis. Find the inverse of the translation matrix.

2. Show that if an $n \times n$ Hermitian matrix has a negative Eigen value, then it cannot be positive definite.

3. On the following basis of $\mathbf{R^4}$, do Gram Schmidt orthogonalization (assuming the standard inner product). $[1, 1, 1, 1], [1, 1, 1, 0], [1, 1, 0, 0], [1, 0, 0, 0]$. Find the matrix of translation from this basis to the standard basis.

4. Consider the symmetric positive definite 2×2 real matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ Find the matrix B such that the above matrix can be written as $B^T B$.

5. Let T be a linear transformation on an inner product space V over $\mathbf{C}$. We define T to be a **Hermitian operator** if $(u, Tv) = (Tu, v)$ for all u, v in V. Let $b_1, b_2, .., b_n$ be an orthonormal basis of V. Prove that the matrix of T is a *Hermitian matrix* with respect to this basis. (Recall that an $n \times n$ complex matrix A is Hermitian if $A^* = A$.) Conversely, prove that if the matrix of an operator with respect to any orthonormal basis is Hermitian, then the operator is also Hermitian.

6. Let T be a linear transformation on an inner product space V over $\mathbf{C}$. We define T to be a **Unitary operator** if $(Tu, Tv) = (u, v)$ for all u, v in V. Let $b_1, b_2, .., b_n$ be an orthonormal basis of V. Prove that the matrix of T is a *Unitary matrix* with respect to this basis. (Recall that an $n \times n$ complex matrix A is Unitary if $A^*A = I$.) Conversely, prove that if the matrix of an operator with respect to any orthonormal basis is Unitary, then the operator is also Unitary. Show that if T is a Unitary operator, the $||T(v)|| = ||v||$ for all $v \in V$.

7. An operator T on an inner product space V of dimension n over $\mathbf{C}$ is said to be orthogonally diagonalizable if there exists an orthonormal basis $b_1, b_2, .., b_n$ such that $b_1, b_2, .., b_n$ are Eigen vectors of T. (Let the corresponding Eigen values be $\lambda_1, \lambda_2, ..., \lambda_n$, not all necessarily distinct.) Let T be orthogonally diagonalizable and let A be the matrix of T with respect to the standard basis. Show that there is a unitary matrix U such that $A = UDU^*$ where U is a unitary $n \times n$ matrix and D a digonal matrix. What can you conclude about the entries of the diagonal matrix D?