Assignment VIII

1. (Revision Question) Let V be an inner product space of dimension n over $\mathbf{C}$. Let W be a subspace of dimension k. Let $b_1, b_2, ..., b_k$ be an orthonormal basis for W. Define the space $W^{\perp} = \{u \in V : (u, w) = 0 \text{ for all } w \in W\}$. Let v be any vector in V. Define the vectors $v_1 = (v, b_1)b_1 + (v, b_2)b_2 + (v, b_k)b_k$ and $v_2 = v - v_1$. Here, v_1 is called the component of v **along** the subspace W and v_2 is called the component of v **orthogonal/perpendicular** to W

   1. Show that $v_2 \in W^{\perp}$. Thus conclude that every vector $v \in V$ can be expressed as a sum $v = v_1 + v_2$ with $v_1 \in W$ and $v_2 \in W^{\perp}$.
   2. Show that $W \cap W^{\perp} = \{0\}$
   3. $dim(W^{\perp}) = n - dim(W)$ (Assume $c_1, c_2, .., c_l$ be a basis of $W^{\perp}$. Show that $c_1, c_2, .., c_l, b_1, b_2, .., b_k$ is a basis of V, thus proving that $l + k = n$ as required).
   4. Show that v_1 and v_2 are uniquely defined. That is, if $v = v_1' + v_2'$ for some $v_1' \in W$ and $v_2' \in W^{\perp}$, then $v_1' = v_1$ and $v_2' = v_2$ (It thus follows that the choice of the particular basis $b_1, b_2, .., b_k$ for W in defining v_1 and v_2 is inconsequential.)
   5. Show that $||v_1||^2 + ||v_2||^2 = ||v||^2$.
   6. If we define orthogonal projection operator P_W on to W as: $P_W(v) = w$, where w is the unique vector in W whose existance was proved above, then show that P_W satisfies the properties 1. $P_W(P_W(v)) = P_W(v)$ for all $v \in V$ (compactly written $P_W^2 = P_W$) 2. P_W is a linear operator in V. 3. For all $u, v \in V$, $(u, P_W(v)) = (P_W(u), v)$
   7. Show that if $w \in W$, then $P_W(w) = w$.
   8. Show if $w' \in W$, and $w' \neq w$ then $d(w, v) < d(w'v)$ (Approximation Theorem)

2. Recall that an operator T on an innner product space V over $\mathbf{C}$ is a **unitary operator** if $(Tu, Tv) = (u, v)$ for all $u, v \in V$. Let $\overline{b} = (b_1, b_2, .., b_n)$ and $\overline{c} = (c_1, c_2, .., c_n)$ be two different orthonormal basis for V. Let A_1 and A_2 be the matrices of T with respect to basis $\overline{b}$ and $\overline{c}$ respectively. Let B be the basis transformation matrix from $\overline{b}$ to $\overline{c}$. (That is, $\overline{b}$=$\overline{c}B$).

   1. Show that the matrix of basis change from basis $\overline{b}$ to $\overline{c}$ is a unitary transformation.
   2. Show that $A_1 A_1^* = I$ and $A_2 A_2^* = I$. That is, A_1 and A_2 must be a **unitary matrices**. (Recall that an $n \times n$ matrix A is called a unitary matrix if $AA^* = I$).
   3. Show that if A is any unitary matrix; the operator determined by A with respect to basis $\overline{b}$ must be unitary. These two exercises show that unitary transformations correspond to unitary matrices and visa versa.
   4. Show that $T(b_1), T(b_2), .., T(b_n)$ is an orthogonal basis of V.
   5. Prove that the $DFT_n = \frac{1}{\sqrt{n}} V_{\overline{\omega}}$ where $\overline{\omega} = (1, \omega_n, \omega_n^2, ..., \omega_n^{n-1})$, ω_n beging a primitve n^{th} root of unity (that is, $\omega_n = e^{\frac{2\pi j}{n}}$) is a unitary transformation.
   6. When $n = 3$, which are the vectors that result out of applying DFT_3 to the standard basis (e_1, e_2, e_3)? That is, find $(DFT_3(e_1), DFT_3(e_2), DFT_3(e_3))$. Repeat with $n = 4$.
   7. If you think of DFT_n as a basis translation from the standard basis to a new basis (this is possible because DFT_n as defined in the previous question is a unitary transformation), what is the new basis (called the **Fourier basis**) to which DFT_n transforms cordinate system from the standard basis? (Hint: That is not exactly $(DFT_n(e_1), DFT_n(e_2), ..., DFT_n(e_n))$, but quite related to this because the transformation is unitary).

3. An operator H on an innner product space V over $\mathbf{C}$ of dimension n is called a **Hermitian Operator** if $(u, Hv) = (Hu, v)$ for all $u, v \in V$. Let $\overline{b} = (b_1, b_2, .., b_n)$ and $\overline{c} = (c_1, c_2, .., c_n)$ be two different orthonormal basis for V. Let A_1 and A_2 be the matrices of H with respect to basis $\overline{b}$ and $\overline{c}$ respectively. Let B be the basis transformation matrix from $\overline{b}$ to $\overline{c}$. (That is, $\overline{b}$=$\overline{c}B$).

1. Show that $A_1^* = A_1$ and $A_2^* = A_2$. That is, A_1 and A_2 must be a **Hermitian matrices**. (Recall that an $n \times n$ matrix A is called a Hermitian matrix if $A^* = A$).
2. Show that if A is any Hermitian matrix; the operator determined by A with respect to basis $\overline{b}$ must be a Hermitian operator. These two exercises show that Hermitian transformations correspond to Hermitian matrices and visa versa.
3. Is DFT_n a Hermitian transformation? If not, what is the property satisfied by DFT_n?

4. A linear operator P on an inner product space V over $\mathbf{C}$ of dimension n is called an **Orthogonal Projection** (Operator) if it satisfies: 1. $P^2 = P$ (that is, $P(P(v)) = P(v)$ for all $v \in V$) and 2. P is a Hermitian operator (that is, for all $u, v \in V$, $(u, Pv) = (Pu, v)$.) Let $U = Nullspace(P)$ and $W = Img(P)$. .
   1. Argue that $U = W^\perp$.
   2. $(I - P)$ is also an orthogonal projection operator (here I is the identify function) with Image $W^\perp$ and Null space W. These exercises show that every orthogonal projection operator defines the perpendicular projection operator into its Image and conversely.

5. Find the matrix of the orthogonal projection operator on to the $x - y$ plance in $\mathbf{R^3}$ with respect to the standard basis.

6. Write down the vectors forming the basis (Fourier basis) to which DFT_2 transforms the standard basis in $\mathbf{C^2}$.

7. Find the matrix of orthogonal projection operator on to the $x - y$ plance in $\mathbf{R^3}$ with respect to the Fourier basis defined by DFT_2.

8. Let V be an inner product space of dimension n over $\mathbf{C}$. Let : $P_1, P_2, .., P_k$ be projection operators in V and let $\lambda_1, \lambda_2, ..., \lambda_k$ be real numbers. Show that $\lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_k P_k$ is a Hermitian operator. The **Spectral Theorem** asserts that the converse of this statement is also true. That is, every Hermitian operator on V can be expressed as a linear combination of Projection operators.