Mid Semester II **Foundations of Computer Science** **Max:30 Marks**

1. Let u, v be Eigen vectors of an operator T on an inner product space V over $\mathbf{C}$. Let λ and μ the corresponding Eigen values. Suppose $\lambda \neq \mu$, show that u, v are linearly independent. [3]

   Soln: Suppose u, v are linearly dependent. Then, there must be some scalar α such that $u = \alpha v$. But then we have $\lambda u = T(u) = T(\alpha v) = \alpha T(v) = \alpha\mu v = \mu\alpha v = \mu u$. This means $(\lambda - \mu)u = 0$. But by definition of an Eigen vector, $u \neq 0$, consequently $\lambda = \mu$, a contradiction.

2. Suppose u, v be orthogonal vectors in an inner product space V over $\mathbf{C}$. Show that u, v are linearly independent. [3]

   Soln: Suppose $\alpha u + \beta v = 0$ for some scalars α, β. Then $0 = (0, u) = (\alpha u + \beta v, u) = \alpha(u, u) + \beta(v, u) = \alpha||u||^2$. Since $||u|| \neq 0$ we have $\alpha = 0$ when $u \neq 0$. Similarly, $v \neq 0 \Rightarrow \beta = 0$.

3. Find an orthonormal basis that diagonalizes the operator in $\mathbf{R^2}$ whose matrix A wrt the standard basis is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. [3]

   Soln: Since the vectors $b_1 = \frac{1}{\sqrt{2}}[1, 1]^T$, $b_2 = \frac{1}{\sqrt{2}}[1, -1]^T$ are orthonormal Eigen vectors of A, with Eigen values 3 and 1, the operator represented by A in the standard basis will have matrix $A' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ with respect to $[b_1, b_2]$.

4. For the operator of the previous question, find a 2×2 orthonormal basis translation matrix B such that the matrix of the operator wrt the basis defined by B is a diagonal matrix. [3]

   Soln: In the above question we have $[e_1, e_2] = [b_1, b_2]\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Hence we have $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

5. Let A be any $n \times n$ real matrix. Show that $A^T A$ is positive definite if and only if A is non-singular. [6]

   Soln: A is non-singular, if and only if $Ax \neq 0$ for all $x \neq 0$, if and only if $||Ax||^2 \neq 0$ for all $x \neq 0$, if and only if $(Ax, Ax) \neq 0$ for all $x \neq 0$, if and only if $x^T(A^T A)x \neq 0$ for all $x \neq 0$ if and only if $A^T A$ is positive definite (by definition).

6. Find a parity check matrix for the code given by the 1×4 generator matrix $G = [0, 1, 1, 0]$ in $\mathbf{F_2^4}$ List all the codewords in the code. [3]

   Since G is a 1×4 matrix, the code defined by G has dimension 1 and contains just $\{0000, 0110\}$ as codewords. Thus, the dual space must be a three dimensional space consisting of 8 vectors, consiting of $\{0000, 0110, 1001, 1111, 1000, 0001, 0111, 1110\}$. The partity check matrix is a generator matrix for this dual space, and is any 3×4 matrix whose rows generate this dual space. One such matrix is $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

7. Recall that for a subspace W of an inner product space V, we define $W^\perp = \{u \in V : (u, w) = 0$ for all $w \in W\}$. Show that $W \cap W^\perp = \{0\}$. [3]

   Soln: Suppose $v \in W \cap W^\perp$. As v is both in W and $W^\perp$, we have $(v, v) = 0$ which can happen if and only if $v = 0$.

8. Consider the basis in $\mathbf{C^3}$ consisting of the vectors $b_1 = [1, 1, 1]$, $b_2 = [1, \omega, \omega^2]$, $b_3 = [1, \omega^2, \omega]$ where $\omega = e^{j\frac{2\pi}{3}}$ (unit vector at 120^o from the positive real line). [3+3]

   1. Is b_1, b_2, b_3 an orthogonal basis? Prove/Disprove.
   2. Find the matrix B for coordinate translation from the standard basis to this basis.

   Soln: Noting that $\overline{\omega} = \omega^2$ and $\overline{\omega^2} = \omega$, it is easy to see that $[1, \omega^2, \omega]\overline{[1, \omega, \omega^2]^T} = 0$ and consequently the basis is orthogonal, but not orthonormal. (The normalizing factor is $\frac{1}{\sqrt{3}}$). The matrix B of basis translation is easy to find for an orthogonal basis translation; $B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$