PreTest III **Foundations of Computer Science** **Max:18 Marks**

Name and Roll No.: ______________________________

1. Suppose $p(x) = (1+x+x^2+x^3+..+x^{n-1})$, $q(x) = (1-x)$. Find the dot product of $DFT_n(p(x))$ and $DFT_n(q(x))$. Clearly justify your answer. [3]

   Soln: $DFT_n(p(x))^T DFT_n(q(x)) = DFT_n(p(x)q(x)) = DFT_n(1-x^n) = 0$.

2. Let T be a Hermitian operator on a finite dimensional complex inner product space V. Let $\lambda_1 \neq \lambda_2$ be distinct Eigen values of T with corresponding Eigen vectors v_1 and v_2. Show that $(v_1, v_2) = 0$. [3]

   Soln: Since T is Hermitian, we have $(v_1, T(v_2)) = (T(v_1), v_2)$. $LS = (v_1, \lambda_2 v_2) = \lambda_2(v_1, v_2)$ as λ_2 is real as T is Hermitian. Similarly, $RS = \lambda_1(v_1, v_2)$. Since $LS = RS$, we have $(v_1, v_2) = 0$ since $\lambda_1 \neq \lambda_2$.

3. Find the matrix (wrt the standard basis) of the orthogonal projection operator along the direction $\frac{1}{\sqrt{3}}[1,1,1]^T$ in $\mathbf{R^3}$ [3]

   Soln: Let $b_1 = \frac{1}{\sqrt{3}}[1,1,1]^T$. Let $W = span(b_1)$. Let b_2, b_3 be chosen as a basis for $W^\perp$ in $\mathbf{R^3}$. Consider the matrix $P = b_1 b_1^T$. Note that $P(b_1) = b_1, P(b_2) = P(b_3) = 0$. For any vector $v = x_1 b_1 + x_2 b_2 + x_3 b_3 \in \mathbf{R^3}$, where x_1, x_2, x_3 are in $\mathbf{R}$. We have $Pv = (b_1 b_1^T)(v) = (b_1 b_1^T)(x_1 b_1)$ (why?) $= x_1 b_1(b_1^T b_1) = x_1 b_1$ which is precisely the component of v along the direction b_1. Thus, the solution is $P = b_1 b_1^T = \frac{1}{3}[1,1,1]^T[1,1,1]$. Note that P is a 3×3 matrix. (Careful study of this example is important because this technique helps us to compute the matrix (w.r.t the standard basis) for the projection operator along any given direction b_1, provided the coordinates of b_1 are known (w.r.t the standard basis)).

4. Find the matrix of the orthogonal projection operator along the plane perpendicular to $\frac{1}{\sqrt{3}}[1,1,1]^T$ in $\mathbf{R^3}$ (wrt the standard basis). [3]

   Soln: If P is the projection along W, then the perpendicular projection along $W^\perp$ must be $I - P$.

5. Suppose P is a projection operator over a finite dimensional complex inner product space V. Suppose P is also a unitary transformation, what can you conclude about P? justify your answer. [3]

   Soln: Since P is a projection, $P^2 = P$. Since P is unitary, $PP^* = I$. Combining the equations we have, $I = PP^* = P^2P^* = P(PP^*) = PI = P$. Thus, P must be the identify transformation.

   [Note: The original post for this solution was incorrect. This is the corrected version]

6. Suppose P is an orthogonal projection into a subspace W of an inner product space V. Let v be any vector in V and w be any vector in W. Show that $(Pv - w, Pv - w) \leq (v - w, v - w)$. [3]

   Soln: Consider the triangle formed by the end points v, Pv and w. If we can establish that $v - Pv$ is perpendicular to $Pv - w$, then $||v-w|^2 = ||(v-Pv)+(Pv-w)||^2 = ||v-Pv||^2 + ||Pv - w||^2 + 2(v-Pv, Pv-w) = ||v-Pv||^2 + ||Pv-w||^2$ and the required inequality follows. The task of proving that $(v-Pv, Pv-w) = 0$ is left as exercise for the final exam!!