Practice Questions

1. Let G be a group and $a \in G$. Let b, b' satisfy $ba = ab = b'a = ab' = e$ where e is the identify. Show that $b = b'$. This shows the inverse of an element is unique in a group. [*]

2. Show that a field is always an integral domain. [*]

3. Show that every finite integral domain is a field. [*]

4. Find all solutions to the equation $49x = 91 \mod 14$. (In general, suppose you are given equation $ax = b \mod n$. with $d = GCD(a, n)$ and b being a multiple of d. Is it true that $\frac{a}{d} = \frac{b}{d} \mod \frac{n}{d}$? - This will help you to solve.) [*]

5. Show that the group $\mathcal{Z}_n \times \mathcal{Z}_m$ is a cyclic group with respect to addition if and only if m and n are co-prime. (In general, show that if C_m and C_n are cyclic groups of order m and n, the product group $C_m \times C_n$ is cyclic if and only if m and n and co-prime. Whenever m, n are co-prime having a and b respectively as generators, show that (a, b) generates $C_m \times C_n$. [*]

6. A number x gives remainder 2 when divided by 3, remainder 3 when divided by 5, remainder 4 when divided by 7 and remainder 7 when divided by 9. Find all possible integer values of x that satisfy this condition using Chinese remainder theorem. [*]

7. Show that the subset of even elements in $\mathcal{Z}_{10}$ is a commutative ring with unity. (What is the unity?). [*]

8. Let $\mathcal{Q}[x]$ be the ring of polynomials with coefficients in Q Consider the map Φ from $\mathcal{Q}[x]$ to Q defined by $\Phi(f) = f(0)$. That is, the map that evaluates the polynomial at zero. Is Φ a homomorphism? What is $ker(\Phi)$? What is $img(\Phi)$? [*]

9. For what integer values of m can there be a ring homomorphism from Z_{10} to Z_m? Justify your answer. [*]

10. Consider the ring $\mathcal{Z}_4 \times \mathcal{Z}_6$ consisting of Consider the ring homomorphism f from $\mathcal{Z}_{24}$ to $\mathcal{Z}_4 \times \mathcal{Z}_6$ sending the element x in $\mathcal{Z}_{24}$ to the tuple $(x \mod 4, x \mod 6)$. The map f is a ring homomorphism. [*]
    - What is $ker(f)$ and $img(f)$?
    - What is $ker(f)$ and $img(f)$ if the map was from $\mathcal{Z}_{15}$ to $\mathcal{Z}_3 \times \mathcal{Z}_5$?